

CLOSED-FORM SOLUTIONS FOR SATELLITE RELATIVE MOTION IN AN AXIALLY-SYMMETRIC GRAVITATIONAL FIELD

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A different approach is proposed for the study of satellite relative motion in an axially-symmetric gravitational field. Instead of using the Keplerian motion as the generating nominal orbit for the absolute motion, another “unperturbed” orbit is proposed instead: An equatorial orbit about an oblate planet. Based on the superintegrability of such motion, closed-form solutions for the equatorial relative motion are obtained. Analytic conditions for the periodicity of the relative motion in a generic central force field are presented, and utilized to design long-term bounded relative motion under high-order even zonal perturbations.

INTRODUCTION

Recent years have seen a growing interest in distributed space systems (DSS) and spacecraft formation flying (SFF).¹ Traditional applications of SFF and related technologies include high-resolution imaging, data relaying, interferometry and in-situ gravitometry. Emerging applications encompass self-assembling satellites, on-orbit replenishment, autonomous collocation and fractionated spacecraft. While traditional SFF systems include multiple cooperative satellites, fractionated spacecraft are multiple free-flying modules interacting through wireless communication links to form a single virtual platform. Each module has one or more pre-determined functions: Navigation, attitude control, power generation or payload operation. The modules do not necessarily have to operate in a tightly-controlled formation, such as in traditional SFF; instead, the modules are required to maintain bounded relative positions, typically less than about 100 kilometers, for the entire mission lifetime. This concept is referred to as *cluster flying*.²

In general, without control forces, two initially close satellites, usually referred to as *chief* and *deputy*, may drift apart due to differential accelerations. Control forces can mitigate this drift, but at a cost of considerable fuel consumption. Moreover, in cluster flying some of the modules may lack propulsive means altogether. It is thus imperative to identify orbits on which the satellites or satellite modules remain within some pre-specified relative distance for the entire mission lifetime. These orbits can considerably reduce the propellant mass required for formationkeeping and render the entire mission much more cost-effective. In the unperturbed two-body problem, such orbits can be found with no difficulties; for instance, one can choose two orbits having the same semimajor axis and slight differences between the other orbital elements. These conditions yield periodic relative motion,³ for which the inter-satellite distances can be calculated.⁴ However, when perturbations are present, the problem becomes more involved.

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The most significant perturbation affecting low Earth-orbit satellites (above an altitude of about 700 km, i. e., almost without drag) is the Earth oblateness, namely the J_2 term in the geopotential. Many works have dealt with modeling relative motion under J_2 . Ross⁵ assumed a fixed circular reference orbit to obtain linearized equations of relative motion. However, since J_2 induces nodal precession and apsidal rotation, the resulting model is only valid for a very short time. Hamel and de Lafontaine⁶ derived linearized equations yielding closed-form solutions for the relative motion. These expressions were written in terms of the chief’s true anomaly and the initial Keplerian elements of the chief and deputy satellites. The J_2 effects were accounted for by using a linear mapping from osculating to mean elements in order to model the differential semimajor axis. Wiesel⁷ provided a representation of spacecraft relative motion about an oblate Earth, assuming near-circular orbits, by utilizing the Floquet theory and perturbation theory. Vadali⁸ developed analytical expressions for the relative motion between J_2 -perturbed satellites utilizing mean classical orbital elements. The resulting theory is usually referred to as the *unit sphere approach*. Schweighart⁹ derived a linearized model for relative motion under J_2 perturbations for satellites moving in almost-circular orbits. Schaub and Alfriend¹⁰ determined constraints leading to perturbation mitigation for relative orbits under the secular effects of J_2 . By using a geometric method, Gim and Alfriend^{11, 12} obtained a state transition matrix for the relative motion between satellites on elliptic orbits subject to J_2 effects.

Higher-order models for J_2 -perturbed motion were suggested as well. Sengupta¹³ derived a second-order model representing the J_2 -perturbed relative motion. Since second-order terms are taken into account, the model can be used to predict the relative dynamics even when the distances between the satellites are relatively large. The reader is referred to^{1, 14} for a comprehensive discussion of satellite relative motion under the effects of J_2 .

Most existing models for J_2 -perturbed satellite relative motion utilize mean orbital elements, linearized local-vertical, local-horizontal (LVLH) dynamics or a combination thereof. Generally speaking, both approaches are limited in precision: Using first-order averaging of the orbital elements omits second-order secular and long-periodic effects ($\mathcal{O}(J_2^2)$); LVLH-based approximations are valid only under particular conditions guaranteeing bounded motion, and hence may be unviable for long-term modeling.

The current work is anchored in the context of long-term satellite relative motion and cluster flying. In particular, the purpose of the present paper is threefold: (i) To develop closed-form solutions for J_2 -perturbed satellite relative motion; (ii) to present necessary and sufficient periodicity conditions for J_2 -perturbed satellite relative motion; and (iii) to extend the aforementioned results to the case of *any* high-order even zonal harmonics (J_{2n} , $n \geq 1$).

The key underlying observation is that even in the presence of J_{2n} , $n \geq 1$ perturbations, there exist a case where the problem of absolute motion is integrable: Equatorial orbits. In fact, in the equatorial case the said problem constitutes a particular case of motion in a central force field, and is therefore *superintegrable* in the sense of Liouville.^{15, 16} The relative motion in the equatorial plane of an oblate – or more generally, an axially-symmetric – planet therefore inherits this superintegrability.

When only J_2 is taken into consideration, there exist an exact closed-form solution for the absolute motion in the equatorial plane, which may be expressed with the help of elliptic integrals.¹⁷ The corresponding closed-form solution for the relative motion is derived in the present paper. Moreover, general necessary and sufficient periodicity conditions for the motion in a central force field are

determined. The boundedness of the relative motion in the equatorial plane of an axially-symmetric planet becomes equivalent to a 1:1 matching of two conserved quantities, which are dependent on the first integrals of the motion.

The main idea when considering inclined orbits, which are no longer integrable (except polar orbits), is to consider a new “unperturbed” absolute motion that incorporates all the terms, including non-Keplerian specific forces, which still preserve integrability. Mathematically, it means that the non-central term only appears in the right-hand side of the equations-of-motion. Theoretically, the new central-force orbit replaces the classical Keplerian ellipse, providing closed-form solutions for both absolute and relative motions. Numerical examples, together with a semi-analytical argumentation based on the mean osculating elements approach, show that the periodicity conditions for the “unperturbed” relative motion are sufficient for boundedness of the relative motion under the presence of arbitrarily high-order zonal harmonics.

The present approach is based neither on linearization nor mean osculating elements; it utilizes analytical tools to investigate the nonlinear model of the perturbed relative motion. Therefore, as subsequently illustrated using numerical simulations, the results are accurate for long periods of time.

PRELIMINARIES

This section presents the general context of the relative motion in an axially-symmetric gravitational potential, together with some useful considerations related to the absolute motion in a central force field.

Motion in a Central Force Field

The absolute motion of a particle in a central force field is a well-studied subject. In the following discussion, some facts that are essential for the present approach are reviewed.

The motion in a central positional force field with respect to an inertial frame originating at the attraction center is governed by the initial value problem (IVP):

$$\left\{ \begin{array}{l} \ddot{\mathbf{r}} + g(r) \hat{\mathbf{r}} = \mathbf{0}, \\ \left\{ \begin{array}{l} \mathbf{r}(t_0) = \mathbf{r}^0, \\ \dot{\mathbf{r}}(t_0) = \mathbf{v}^0, \end{array} \right. \end{array} \right. \quad (1)$$

where $\mathbf{r} \in \mathbb{R}^3$ is the position vector, and g is a continuous scalar function of the positive scalar variable $r = \|\mathbf{r}\|$. This problem is *maximally superintegrable* in the sense of Liouville, since it possesses the following five independent integrals of motion: the specific angular momentum,

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}, \quad (2)$$

the specific energy,

$$\mathcal{E} = \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + V(r), \quad (3)$$

where $V = V(r)$ is the specific potential energy, defined by:

$$V(r) = \int g(r) dr, \quad (4)$$

and a Laplace-Runge-Lenz-like first integral,^{15,18} usually introduced as a unit vector lying in the plane of motion.

If the initial conditions are chosen so that the motion described by (1) is bounded, and there are no bifurcations in the phase plane (r, \dot{r}) , and the phase-space orbit is closed and stable, then the trajectory is contained inside an annulus defined by two concentric circles of radii r_{\min} and r_{\max} , as shown in Fig. 1. By introducing the effective potential energy:¹⁹

$$U(r) = \frac{h^2}{2r^2} + V(r), \quad (5)$$

where $h = \|\mathbf{h}\|$, the constants r_{\min} and r_{\max} are two of the solutions of the algebraic equation*:^{16,19}

$$\mathcal{E} - U(r) = 0. \quad (6)$$

The time equation for the motion in a central force field is given by:¹⁹

$$(t - t_0) \bmod \frac{T}{2} = \int_{r^0}^r \frac{1}{\sqrt{2(\mathcal{E} - U(s))}} ds. \quad (7)$$

where $r(t_0) = r^0$ and the relationship between the polar angle and the distance is given by:

$$[\theta(t) - \theta(t_0)] \bmod \varphi = \int_{r^0}^r \frac{hs^{-2}}{\sqrt{2(\mathcal{E} - U(s))}} ds. \quad (8)$$

In (7) and (8), the following constants of motion are introduced:¹⁹ the *radial period*, T ,

$$T \triangleq 2 \int_{r_{\min}}^{r_{\max}} \frac{1}{\sqrt{2(\mathcal{E} - U(s))}} ds \quad (9)$$

and the *orbital angle*, φ ,

$$\varphi \triangleq \int_{r_{\min}}^{r_{\max}} \frac{hs^{-2}}{\sqrt{2(\mathcal{E} - U(s))}} ds, \quad (10)$$

as shown in Fig. 1. In (9), T represents the main period of $r(t)$. The quantities appearing in (9) and (10) are well defined if and only if r_{\min} and r_{\max} are simple zeros of the denominator, i.e.:

$$\left\{ \begin{array}{l} \mathcal{E} - U(s) = 0, s \in \{r_{\min}, r_{\max}\}; \\ \left. \frac{d[\mathcal{E} - U(s)]}{ds} \right|_{s \in \{r_{\min}, r_{\max}\}} \neq 0. \end{array} \right. \quad (11)$$

*This happens when the effective potential energy has a critical point that is a minimum, and the initial conditions are adequately chosen.

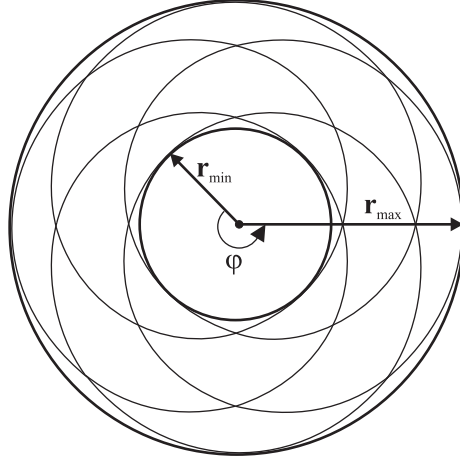


Figure 1. Motion in a central force field

Absolute Motion in an Axially-Symmetric Gravitational Field

The motion of a particle in the gravitational potential field generated by a fixed-shaped homogeneous body is modeled by the IVP:

$$\ddot{\mathbf{r}} = -\frac{\partial V}{\partial \mathbf{r}}, \quad \begin{cases} \mathbf{r}(t_0) = \mathbf{r}^0, \\ \dot{\mathbf{r}}(t_0) = \mathbf{v}^0, \end{cases} \quad (12)$$

where $\mathbf{r} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is the position vector in an inertial reference frame with the origin at the center of mass of the attractive body, $t_0 \geq 0$ is the initial time and $V = V(\mathbf{r})$ is the gravitational potential, given by:²⁰

$$V(\mathbf{r}) = \frac{\mu}{M} \int_{Q \in \mathcal{D}} \frac{dm(\mathbf{r})}{\|\mathbf{r} - \mathbf{r}_Q\|}, \quad (13)$$

with μ being the gravitational parameter, M the mass of the attraction body, \mathcal{D} denotes the spatial domain occupied by the attractive body and \mathbf{r}_Q is the position vector of a generic point Q of the attractive body with respect to the same inertial reference frame originating at its center of mass.

If the attracting body has a rotational symmetry, then the potential expressed in Eq. (13) may be expanded into a series of the form:

$$V(\mathbf{r}) = -\frac{\mu}{r} \left\{ 1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_{eq}}{r} \right)^n P_n(\cos \phi) \right\}, \quad (14)$$

where r_{eq} is the equatorial radius of the attracting body, J_n are the zonal harmonics coefficients, P_n are the Legendre polynomials, and ϕ is the colatitude angle, defined as²¹:

$$\cos \phi = \frac{\mathbf{r} \cdot \mathbf{i}_z}{r}, \quad (15)$$

where \mathbf{i}_z is a unit vector associated with the direction of the symmetry axis of the attracting body.

The integrability of the IVP (12) with the potential having the form (14) is still an open problem. It was proven that when only the J_2 term is retained, the problem is generally non-integrable.^{22,23}

The Legendre Polynomials in the Gravitational Potential

We present here some useful properties of the Legendre polynomials, to be utilized in the paper. The well-known Rodrigues formula:²⁴

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n] \quad (16)$$

leads to:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_n^k C_{2n-2k}^n x^{n-2k}, \quad (17)$$

where $\lfloor y \rfloor$ is the floor function, and:

$$C_n^k \triangleq \frac{n!}{k!(n-k)!}. \quad (18)$$

From (17), it follows that if the degree n of the Legendre polynomial P_n is even, then P_n contains a free term. If n is odd, P_n has no free terms. The free term of P_{2n} is:

$$P_{2n}(0) = \frac{(-1)^n C_{2n}^n}{2^{2n}}, \quad n \geq 1. \quad (19)$$

Consequently, the potential expressed in (14) may be separated as follows:

$$V(\mathbf{r}) = -\frac{\mu}{r} \{1 - V_c(r) - V_p(\mathbf{r})\}, \quad (20)$$

where:

$$V_c(r) = \sum_{n \geq 1} \frac{(-1)^n J_{2n} C_{2n}^n}{2^{2n}} \left(\frac{r_{eq}}{r}\right)^{2n}, \quad (21)$$

$$V_p(\mathbf{r}) = V_p(r, \cos \phi) = -V_c(r) + \sum_{n=2}^{\infty} J_n \left(\frac{r_{eq}}{r}\right)^n P_n(\cos \phi). \quad (22)$$

Note that $V_p(\mathbf{r})$ does not contain any terms that are dependent only upon the magnitude of the position vector r . In particular,

$$V_c(r) = -\frac{J_2}{2} \left(\frac{r_{eq}}{r}\right)^2 + \frac{3J_4}{8} \left(\frac{r_{eq}}{r}\right)^4 - \frac{5J_6}{16} \left(\frac{r_{eq}}{r}\right)^6 + \frac{35J_8}{128} \left(\frac{r_{eq}}{r}\right)^8 - \frac{J_{10}}{4} \left(\frac{r_{eq}}{r}\right)^{10} + \dots \quad (23)$$

$$\begin{aligned} V_p(r, \cos \phi) = & \frac{3J_2}{2} \left(\frac{r_{eq}}{r}\right)^2 \cos^2 \phi + \frac{J_3}{2} \left(\frac{r_{eq}}{r}\right)^3 (5 \cos^3 \phi - 3 \cos \phi) \\ & + \frac{J_4}{8} \left(\frac{r_{eq}}{r}\right)^4 (35 \cos^4 \phi - 30 \cos^2 \phi) + \\ & + \frac{J_5}{8} \left(\frac{r_{eq}}{r}\right)^5 (63 \cos^5 \phi - 70 \cos^3 \phi + 15 \cos \phi) + \\ & + \frac{J_6}{16} \left(\frac{r_{eq}}{r}\right)^6 (231 \cos^6 \phi - 315 \cos^4 \phi + 105 \cos^2 \phi) + \dots \end{aligned} \quad (24)$$

Relative Motion in an Axially-Symmetric Gravitational Field

Consider two satellites, chief and deputy, orbiting an attraction center that generates an axially-symmetric gravitational potential field, $V = V(\mathbf{r})$. Let $\mathbf{r}_0, \dot{\mathbf{r}}_0$ denote the respective inertial position and velocity vectors of the chief, and $\mathbf{r}_1, \dot{\mathbf{r}}_1$ be the same quantities related to the deputy. The motion of the deputy with respect to the usual local-vertical local-horizontal (LVLH) frame²¹ attached to the mass center of the chief satellite is described by the IVP:

$$\begin{cases} \ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega}_0 \times \dot{\boldsymbol{\rho}} + \boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{\rho} = - \left[\frac{\partial V(\boldsymbol{\rho} + \mathbf{r}_0)}{\partial(\boldsymbol{\rho} + \mathbf{r}_0)} - \frac{\partial V(\mathbf{r}_0)}{\partial \mathbf{r}_0} \right], \\ \begin{cases} \boldsymbol{\rho}(t_0) = \Delta \boldsymbol{\rho}, \\ \dot{\boldsymbol{\rho}}(t_0) = \Delta \mathbf{v}, \end{cases} \end{cases} \quad (25)$$

where $(\boldsymbol{\rho}, \dot{\boldsymbol{\rho}}) \in \mathbb{R}^3 \times \mathbb{R}^3$ are the position and velocity vectors of the deputy in the LVLH frame, $\boldsymbol{\omega}_0 \in \mathbb{R}^3$ is the angular velocity of the LVLH frame with respect to an inertial frame (expressed in the LVLH frame), and $-\mathbf{r}_0$ models the motion of the attraction center relative to the LVLH frame.

When taking into consideration the expression (20) for the potential V , the IVP (25) may be rewritten into:

$$\begin{cases} \ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega}_0 \times \dot{\boldsymbol{\rho}} + \boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{\rho} + g(\|\boldsymbol{\rho} + \mathbf{r}_0\|) \frac{(\boldsymbol{\rho} + \mathbf{r}_0)}{\|\boldsymbol{\rho} + \mathbf{r}_0\|} \\ -g(r_0) \frac{\mathbf{r}_0}{r_0} = - \left[\frac{\partial V_p(\boldsymbol{\rho} + \mathbf{r}_0)}{\partial(\boldsymbol{\rho} + \mathbf{r}_0)} - \frac{\partial V_p(\mathbf{r}_0)}{\partial \mathbf{r}_0} \right], \\ \begin{cases} \boldsymbol{\rho}(t_0) = \Delta \boldsymbol{\rho}, \\ \dot{\boldsymbol{\rho}}(t_0) = \Delta \mathbf{v}, \end{cases} \end{cases} \quad (26)$$

where g is a continuous scalar-valued function defined as:

$$g(r) \triangleq \frac{\partial V_c}{\partial \mathbf{r}} \cdot \hat{\mathbf{r}} \quad (27)$$

and V_c, V_p are defined in (21) and (22), respectively. Here $\hat{\mathbf{u}}$ denotes the unit vector associated to vector \mathbf{u} .

The right-hand side of (26) may be regarded as a perturbation of the dynamics described by the IVP:

$$\begin{cases} \ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega}_0 \times \dot{\boldsymbol{\rho}} + \boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}}_0 \times \boldsymbol{\rho} + g(\|\boldsymbol{\rho} + \mathbf{r}_0\|) \frac{(\boldsymbol{\rho} + \mathbf{r}_0)}{\|\boldsymbol{\rho} + \mathbf{r}_0\|} - g(r_0) \hat{\mathbf{r}}_0 = \mathbf{0}, \\ \begin{cases} \boldsymbol{\rho}(t_0) = \Delta \boldsymbol{\rho}, \\ \dot{\boldsymbol{\rho}}(t_0) = \Delta \mathbf{v}, \end{cases} \end{cases} \quad (28)$$

which is the relative motion equation in a central force field, with \mathbf{r}_0 determined by the chief's dynamics, given by:²⁵

$$\begin{cases} \ddot{\mathbf{r}}_0 + 2\boldsymbol{\omega}_0 \times \dot{\mathbf{r}}_0 + \boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \mathbf{r}_0) + \dot{\boldsymbol{\omega}}_0 \times \mathbf{r}_0 + g(r_0) \hat{\mathbf{r}}_0 = \mathbf{0}, \\ \begin{cases} \mathbf{r}_0(t_0) = \mathbf{r}_0^0, \\ \dot{\mathbf{r}}_0(t_0) = \mathbf{v}_0^0. \end{cases} \end{cases} \quad (29)$$

If the initial conditions are chosen so that:

$$(\mathbf{r}_0^0 \times \mathbf{v}_0^0) \times \mathbf{i}_z = \mathbf{0}, \quad \Delta \boldsymbol{\rho} \cdot \boldsymbol{\omega}_0 = 0, \quad \Delta \mathbf{v} \cdot \boldsymbol{\omega}_0 = 0, \quad (30)$$

then Equations (28), (29) model the relative motion in the equatorial plane under the influence of any potential field with two orthogonal axes of symmetry. In this case, $\boldsymbol{\omega}_0$ has a fixed direction – that of the constant angular momentum \mathbf{h}_0 of the chief's inertial orbit, i. e.

$$\boldsymbol{\omega}_0 = \frac{\mathbf{h}_0}{r_0^2}. \quad (31)$$

CLOSED-FORM SOLUTION TO THE PROBLEM OF RELATIVE MOTION IN A CENTRAL FORCE FIELD

As noted in the previous section, the relative motion in an axially-symmetric gravitational field may be regarded as a perturbation of the relative motion in a central force field, described by the IVP (28). To see this, consider the Lie group $SO(3)$, i.e., the group of real 3×3 orthogonal matrices with determinant equal to 1. Let $\mathfrak{so}(3)$ denote its associated Lie algebra. Recall that the elements of $\mathfrak{so}(3)$ are 3×3 skew-symmetric matrices. The map $\tilde{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ denotes the usual Lie algebra isomorphism that identifies $\mathfrak{so}(3)$ and the matrix commutator bracket, with \mathbb{R}^3 and the vector cross product,

$$\mathbf{w} = [w_1, w_2, w_3]^T \mapsto \tilde{\mathbf{w}} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}. \quad (32)$$

The solution to the IVP (28) is:^{26,27}

$$\boldsymbol{\rho}(t) = \mathbf{R}_{-\boldsymbol{\omega}_0} \mathbf{r}_1(t) - r_0(t) \hat{\mathbf{r}}_0^0, \quad (33)$$

where $\mathbf{R}_{-\boldsymbol{\omega}_0} \in SO(3)$ is the solution to the IVP:²⁵

$$\dot{\mathbf{R}}_{-\boldsymbol{\omega}_0} = -\tilde{\boldsymbol{\omega}_0} \mathbf{R}_{-\boldsymbol{\omega}_0}, \quad \mathbf{R}_{-\boldsymbol{\omega}_0}(t_0) = \mathbf{I}_3, \quad (34)$$

\mathbf{r}_1 is the solution to the IVP:^{26,27}

$$\left\{ \begin{array}{l} \ddot{\mathbf{r}}_1 + g(r_1) \hat{\mathbf{r}}_1 = \mathbf{0}, \\ \left\{ \begin{array}{l} \mathbf{r}_1(t_0) = \mathbf{r}_0^0 + \Delta \boldsymbol{\rho}, \\ \dot{\mathbf{r}}_1(t_0) = \mathbf{v}_0^0 + \Delta \mathbf{v} + \boldsymbol{\omega}_0(t_0) \times \Delta \boldsymbol{\rho}. \end{array} \right. \end{array} \right. \quad (35)$$

and r_0 is the magnitude of the solution to the IVP (29), and is also the solution of the IVP:

$$\left\{ \begin{array}{l} \ddot{r}_0 - \frac{h_0^2}{r_0^3} + g(r_0) = 0, \\ \left\{ \begin{array}{l} r_0(t_0) = r_0^0, \\ \dot{r}_0(t_0) = \mathbf{v}_0^0 \cdot \hat{\mathbf{r}}_0^0. \end{array} \right. \end{array} \right. \quad (36)$$

The closed-form expression for $\mathbf{R}_{-\boldsymbol{\omega}_0}$ is:²⁵

$$\mathbf{R}_{-\boldsymbol{\omega}_0} = \mathbf{I}_3 - \sin \theta_0(t) \frac{\tilde{\mathbf{h}}_0}{h_0} + [1 - \cos \theta_0(t)] \left(\frac{\tilde{\mathbf{h}}_0}{h_0} \right)^2. \quad (37)$$

In (37), θ_0 denotes the polar angle of the chief, measured from the reference direction defined by $\hat{\mathbf{r}}_0^0$ (the polar angle of the deputy, measured from $\hat{\mathbf{r}}_1^0$, is denoted by θ_1). The angles $\theta_k = \theta_k(t_0, t)$ satisfy:

$$\theta_k = h_k \int_{t_0}^t \frac{1}{r_k^2(s)} ds, k = 0, 1. \quad (38)$$

The relative velocity may be computed from:²⁷

$$\dot{\boldsymbol{\rho}}(t) = \mathbf{R}_{-\omega_0} [\dot{\mathbf{r}}_1(t) - \tilde{\boldsymbol{\omega}}_0 \mathbf{r}_1(t)] - \dot{r}_0(t) \hat{\mathbf{r}}_0^0. \quad (39)$$

Equation (33) can also be written as:

$$\boldsymbol{\rho}(t) = r_1(t) \mathbf{R}_{\Delta\omega} \hat{\mathbf{r}}_1^0 - r_0(t) \hat{\mathbf{r}}_0^0, \quad (40)$$

where:

$$\mathbf{R}_{\Delta\omega} = \mathbf{R}_{-\omega_0} \mathbf{R}_{\omega_1}, \quad (41)$$

and:

$$\mathbf{R}_{\omega_1} = \mathbf{I}_3 + \sin \theta_1(t) \frac{\tilde{\mathbf{h}}_1}{h_1} + [1 - \cos \theta_1(t)] \left(\frac{\tilde{\mathbf{h}}_1}{h_1} \right)^2. \quad (42)$$

Coplanar Absolute Orbits

If the two absolute orbits of the satellites lie in the same plane, then the expression for $\mathbf{R}_{\Delta\omega}$ can be simplified,

$$\mathbf{R}_{\Delta\omega} = \mathbf{I}_3 + \sin \beta(t) \frac{\tilde{\mathbf{h}}_0}{h_0} + [1 - \cos \beta(t)] \left(\frac{\tilde{\mathbf{h}}_0}{h_0} \right)^2, \quad (43)$$

with $\beta(t)$ is expressed as:

$$\beta(t) = \theta_1(t) - \theta_0(t) + \delta, \quad (44)$$

where δ is the angle between $\hat{\mathbf{r}}_0^0$ and $\hat{\mathbf{r}}_1^0$, as shown in Fig. 2).

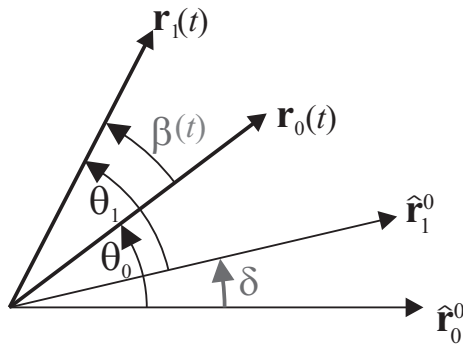


Figure 2. Angles in the coplanar relative motion

Writing $\boldsymbol{\rho} = [\rho_x, \rho_y, \rho_z]^T$, it follows that $\rho_z = 0$, and (40) becomes

$$\begin{bmatrix} \rho_x \\ \rho_y \end{bmatrix} = r_1 \begin{bmatrix} \cos \beta(t) \\ \sin \beta(t) \end{bmatrix} - r_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (45)$$

which yields:

$$\begin{bmatrix} \dot{\rho}_x \\ \dot{\rho}_y \end{bmatrix} = \dot{r}_1 \begin{bmatrix} \cos \beta(t) \\ \sin \beta(t) \end{bmatrix} + r_1 \left(\frac{h_1}{r_1^2} - \frac{h_0}{r_0^2} \right) \begin{bmatrix} -\sin \beta(t) \\ \cos \beta(t) \end{bmatrix} - \dot{r}_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (46)$$

The relative velocity can be also expressed as:

$$\begin{bmatrix} \dot{\rho}_x \\ \dot{\rho}_y \end{bmatrix} = A(t) \begin{bmatrix} \cos [\beta(t) + \sigma(t)] \\ \sin [\beta(t) + \sigma(t)] \end{bmatrix} - \dot{r}_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (47)$$

where $A(t)$, $\sigma(t)$ are defined as:

$$A(t) = \sqrt{\dot{r}_1^2 + r_1^2 \left(\frac{h_1}{r_1^2} - \frac{h_0}{r_0^2} \right)^2}, \quad (48)$$

$$\begin{cases} \sin \sigma(t) = \frac{r_1}{A(t)} \left(\frac{h_1}{r_1^2} - \frac{h_0}{r_0^2} \right); \\ \cos \sigma(t) = \frac{\dot{r}_1}{A(t)}. \end{cases} \quad (49)$$

Closed-Form Solution for J_2 -perturbed Relative Motion in the Equatorial Plane

When only the J_2 term in the geopotential is taken into account, and the motion is confined to the equatorial plane, the specific potential energy $V(\mathbf{r})$ in (12) satisfies (cf. also (23)):

$$V(\mathbf{r}) = V_c(r) = -\frac{\mu}{r} \left[1 + \frac{J_2}{2} \left(\frac{r_{eq}}{r} \right)^2 \right]. \quad (50)$$

Substituting (50) into the expression for the effective potential energy (5) and writing down (6) with the total energy as in (3) (only the case $\mathcal{E} < 0$ is treated) yields the following cubic equation for the relative equilibria:

$$r^3 - \frac{\mu}{|\mathcal{E}|} r^2 + \frac{h^2}{2|\mathcal{E}|} r - \frac{\mu J_2 r_{eq}^2}{2|\mathcal{E}|} = 0. \quad (51)$$

In order to have three real and positive solutions to (51), denoted by $0 < r_* < r_{\min} < r_{\max}$, the following conditions must be satisfied:

$$4 \left(\frac{\mu^2}{|\mathcal{E}|} - \frac{3}{2} h^2 \right)^3 > -27 \mu J_2 r_{eq}^2 |\mathcal{E}|^2 + 9 \mu h^2 |\mathcal{E}| - 2 \mu^3, \quad (52)$$

$$2 \mu^2 - 3 h^2 |\mathcal{E}| > 0. \quad (53)$$

By making some computations using Vieta's formulae for the cubic equation (51), the following relations are obtained:

$$\begin{cases} \frac{r_{\min} r_{\max} (r_{\min} + r_{\max})}{r_{\min} r_{\max} - J} = \frac{\mu}{|\mathcal{E}|}; \\ J \frac{(r_{\min} + r_{\max})^2}{r_{\min} r_{\max} - J} + r_{\min} r_{\max} = \frac{h^2}{2|\mathcal{E}|}; \\ r_* = J \frac{r_{\min} + r_{\max}}{r_{\min} r_{\max} - J}, \end{cases} \quad (54)$$

where:

$$J = \frac{J_2 r_{eq}^2}{2}. \quad (55)$$

The orbital period (cf. (9)) is computed from:

$$T = \frac{2}{\sqrt{2|\mathcal{E}|}} \int_{r_{\min}}^{r_{\max}} \frac{s^{\frac{3}{2}}}{\sqrt{(s-r_*)(s-r_{\min})(r_{\max}-s)}} ds, \quad (56)$$

and the orbital angle (cf. (10)) is:

$$\varphi = \frac{h}{\sqrt{2|\mathcal{E}|}} \int_{r_{\min}}^{r_{\max}} \frac{s^{-\frac{1}{2}}}{\sqrt{(s-r_*)(s-r_{\min})(r_{\max}-s)}} ds \quad (57)$$

To proceed, denote by $E(\cdot, \cdot)$, $F(\cdot, \cdot)$, $P(\cdot, \cdot, \cdot)$ the incomplete elliptic integrals of the first, second and third kind, respectively. Then $E(1, \cdot)$, $F(1, \cdot)$, $P(1, \cdot, \cdot)$ will represent the associated complete elliptic integrals. The following expression is obtained for the radial period (see also¹⁷):

$$T = \frac{2}{\sqrt{2|\mathcal{E}|}} \left[C_1 E(1, w) + C_2 F(1, w) + C_3 P\left(1, 1 - \frac{r_{\max}}{r_{\min}}, w\right) \right], \quad (58)$$

where:

$$w = \sqrt{\frac{(r_{\max} - r_{\min}) r_*}{(r_{\max} - r_*) r_{\min}}}, \quad (59)$$

$$C_1 = -\frac{r_{\max} \sqrt{r_{\min}}}{\sqrt{r_{\max} - r_*}}; C_2 = -\sqrt{r_{\min}} \sqrt{r_{\max} - r_*}; \quad (60)$$

$$C_3 = \frac{\mu r_{\max}}{2|\mathcal{E}| \sqrt{(r_{\max} - r_*) r_{\min}}}.$$

By taking into account (54), the following expressions may be derived:

$$C_1 C_2 = r_{\min} r_{\max}; C_3 = -\frac{\mu C_1}{2|\mathcal{E}| r_{\min}}. \quad (61)$$

The orbital angle φ is computed from:

$$\varphi = \frac{2hE(1, w)}{|C_2| \sqrt{2|\mathcal{E}|}}. \quad (62)$$

Consider now the relative motion of two satellites, chief and deputy, in the equatorial plane. The closed-form expression for the relative position in the chief's LVLH frame is given by (33), with the Cartesian correspondence in (45). The absolute distances $r_{0,1}$ will be computed based on (7) and (8):

$$(t - t_0) \bmod \frac{T_k}{2} = \frac{1}{\sqrt{2|\mathcal{E}_k|}} \int_{r_k^0}^{r_k} \frac{s^{\frac{3}{2}} ds}{\sqrt{(s - r_*^{(k)}) (s - r_{\min}^{(k)}) (r_{\max}^{(k)} - s)}}, \quad k = 0, 1, \quad (63)$$

which yields:

$$(t - t_0) \bmod \frac{T_k}{2} = \frac{1}{\sqrt{2|\mathcal{E}_k|}} \left[\sqrt{\frac{\mathcal{E} - U(r_k)}{r_k}} + C_1^{(k)} E(\chi_k(r_k), w_k) + C_2^{(k)} F(\chi_k(r_k), w_k) + C_3^{(k)} P\left(\chi_k(r_k), 1 - \frac{r_{\max}^{(k)}}{r_{\min}^{(k)}}, w_k\right) \right] \Bigg|_{r_k}^{r_k^0}, \quad k = 0, 1, \quad (64)$$

where χ_k is defined as:

$$\chi_k(r_k) = \sqrt{\frac{(r_{\max}^{(k)} - r_k) r_{\min}^{(k)}}{(r_{\max}^{(k)} - r_{\min}^{(k)}) r_k}}, \quad k = 0, 1. \quad (65)$$

Note that if one defines the real-valued function:

$$f(s) = \sqrt{\frac{r_{\max}}{s} - 1}, \quad (66)$$

with the appropriate domain restrictions ($s \neq 0, s < r_{\max}$), then:

$$\begin{cases} w_k = \frac{f(r_{\min}^{(k)})}{f(r_*^{(k)})} \\ \chi_k = \frac{f(r_k)}{f(r_{\min}^{(k)})} \end{cases}, \quad k = 0, 1. \quad (67)$$

The polar angles $\theta_{0,1}$ are computed from:

$$\theta_k \bmod \varphi_k = \frac{h_k}{\sqrt{2|\mathcal{E}_k|}} \int_{r_k^0}^{r_k} \frac{s^{-\frac{1}{2}}}{\sqrt{(s - r_*^{(k)}) (s - r_{\min}^{(k)}) (r_{\max}^{(k)} - s)}} ds, \quad k = 0, 1, \quad (68)$$

and the following expressions are obtained:

$$\theta_k \bmod \varphi_k = \frac{2h_k}{|C_2| \sqrt{2|\mathcal{E}_k|}} E(\chi_k(r_k), w_k) \Bigg|_{r_k}^{r_k^0}, \quad k = 0, 1. \quad (69)$$

From (69), it follows that the distances to the attraction center of the chief and deputy can be written as functions of the polar angles by using the Jacobi elliptic function sn (see also¹⁷):

$$r_k = \frac{r_{\min}^{(k)} r_{\max}^{(k)}}{r_{\min}^{(k)} + (r_{\max}^{(k)} - r_{\min}^{(k)}) \zeta_k^2}, \quad k = 0, 1, \quad (70)$$

where:

$$\zeta_k = \operatorname{sn} \left[E(\chi_k(r_k^0), w_k) - \frac{|C_2^{(k)}| \sqrt{2|\mathcal{E}_k|}}{2h_k} \theta_k, w_k \right]. \quad (71)$$

By denoting:

$$a_k^* \triangleq \frac{r_{\max}^{(k)} + r_{\min}^{(k)}}{2}; \quad e_k^* \triangleq \frac{r_{\max}^{(k)} - r_{\min}^{(k)}}{r_{\max}^{(k)} + r_{\min}^{(k)}}; \quad p_k^* \triangleq a_k^* \left[1 - (e_k^*)^2 \right], \quad k = 0, 1, \quad (72)$$

the expression (70) of the radial distance becomes:

$$r_k = \frac{p_k^*}{1 + e_k^* (2\zeta_k^2 - 1)}, \quad k = 0, 1, \quad (73)$$

which resembles the classical unperturbed Kepler motion. In fact, when setting $J_2 = 0$, (73) provides the well-known Keplerian conic-sections equation:

$$r_k = \frac{p_k}{1 + e_k \cos \theta_k}, \quad k = 0, 1, \quad (74)$$

where e , a , $p = a(1 - e^2)$ denote the eccentricity, semimajor axis and semilatus rectum, respectively. However, the elements a^* and e^* are constant in the J_2 -perturbed equatorial motion, and are therefore different from their osculating counterparts. The magnitude of the relative position vector can now be obtained by writing:

$$\rho = \sqrt{r_0^2 + r_1^2 - 2r_0 r_1 \cos \beta(t)}, \quad (75)$$

where $\beta(t)$ is defined in (44) and $r_{0,1}$, $\theta_{0,1}$ are computed in (64), (69).

Generalization to the J_{2n} Case

The above methodology can be extended to include higher-order even zonal harmonics, J_{2n} , $n \geq 1$. In this case, the equatorial motion is still maximally superintegrable, and thus solvable by quadratures. The expressions (45) for the relative position and (46) for the relative velocity remain exactly the same. Only the expressions for the absolute distance r_k and polar angles θ_k , $k = 0, 1$, will be different. The polynomial that provides $r_{\min}^{(k)}$ and $r_{\max}^{(k)}$ will have a degree $2n + 1$, $n \geq 1$. The algebraic equation to be solved in order to determine these quantities is (cf. (23)):

$$r^{2n+1} - \frac{\mu}{|\mathcal{E}_k|} r^{2n} + \frac{h_k^2}{2|\mathcal{E}_k|} r^{2n-1} - \frac{\mu J_2 r_{eq}^2}{2|\mathcal{E}_k|} r^{2n-2} + \frac{3\mu J_4 r_{eq}^4}{8|\mathcal{E}_k|} r^{2n-3} - \dots = 0, \quad k = 0, 1. \quad (76)$$

The radial period T and the orbital angle φ will be computed as follows:

$$T_k = \frac{2}{\sqrt{2|\mathcal{E}_k|}} \int_{r_{\min}^{(k)}}^{r_{\max}^{(k)}} \frac{s^{\frac{3}{2}}}{\sqrt{(s - r_*^{(k)}) (s - r_{\min}^{(k)}) (r_{\max}^{(k)} - s) Q_k(s)}} ds, \quad k = 0, 1; \quad (77)$$

$$\varphi_k = \frac{h_k}{\sqrt{2|\mathcal{E}_k|}} \int_{r_{\min}^{(k)}}^{r_{\max}^{(k)}} \frac{s^{-\frac{1}{2}}}{\sqrt{(s - r_*^{(k)}) (s - r_{\min}^{(k)}) (r_{\max}^{(k)} - s) Q_k(s)}} ds, \quad k = 0, 1, \quad (78)$$

where $Q_k(s)$ are polynomials of degree $2n - 2$ and $r_*^{(k)}$ is another real root of the polynomial equation (76). The radial motion and the polar angle are determined from:

$$(t - t_0) \bmod \frac{T_k}{2} = \frac{1}{\sqrt{2|\mathcal{E}_k|}} \int_{r_k^0}^{r_k} \frac{s^{\frac{3}{2}} ds}{\sqrt{(s - r_*^{(k)}) (s - r_{\min}^{(k)}) (r_{\max}^{(k)} - s) Q_k(s)}}, \quad k = 0, 1; \quad (79)$$

$$\theta_k \bmod \varphi_k = \frac{h_k}{\sqrt{2|\mathcal{E}_k|}} \int_{r_k^0}^{r_k} \frac{s^{-\frac{1}{2}} ds}{\sqrt{(s - r_*^{(k)}) (s - r_{\min}^{(k)}) (r_{\max}^{(k)} - s) Q_k(s)}}, \quad k = 0, 1. \quad (80)$$

PERIODICITY CONDITIONS FOR RELATIVE MOTION IN A CENTRAL FORCE FIELD

In the previous section, closed-form solutions for equatorial relative motion under the effect of J_2 were derived, and an extension of the methodology to even zonal harmonics was provided. An important question related to the above findings is the existence of periodic equatorial relative orbits in the presence of even zonal harmonics, or more generally, any central force.

Theorem 1 *Consider two particles orbiting a common center on bounded coplanar trajectories, with the gravitational potential being $V = V(r)$. Then the motion of one particle with respect to the LVLH frame attached to the other particle is periodic if and only if the following conditions are satisfied:*

1. *The radial periods T_0 and T_1 are commensurable, i.e. there exist two natural relative prime numbers m, n such that:*

$$nT_0 = mT_1 \triangleq T, \quad (81)$$

2. *The difference between φ_1/T_1 and φ_0/T_0 , is commensurable with π/T :*

$$\left(\frac{\varphi_1}{T_1} - \frac{\varphi_0}{T_0} \right) \left(\frac{\pi}{T} \right)^{-1} \in \mathbb{Q}, \quad (82)$$

where \mathbb{Q} denotes the set of rational numbers.

The proof is omitted for the sake of brevity.

Corollary 1 *The 1:1 commensurability occurs when:*

$$T_0 = T_1; \quad \varphi_0 = \varphi_1. \quad (83)$$

In this case, the deputy spacecraft will have the same position vector in LVLH, as well as the same velocity, after exactly one radial period from the initial epoch. The relative motion will be periodic.

Generalization to Non-equatorial Orbits

The previous section provided a rigorous method for generating periodic relative motion under the influence of J_2 , when both chief and deputy orbits lie in the equatorial plane. In the subsequent discussion, it is shown how to extend these results for generating bounded (quasiperiodic) relative orbits in the non-equatorial case. The proposed method generates bounded orbits not only for an oblate planet (i.e., J_2 only), but for high-order zonal harmonics as well. This statement will be verified numerically further within the present paper.

The main idea is that a new “unperturbed” absolute motion is considered, which takes place in a central force field incorporating all the central terms (i.e., terms depending on the magnitude of the position vector only). The new “unperturbed” absolute motion still remains superintegrable, and so is the relative “unperturbed” motion. In this way, a large part of the perturbation of the Keplerian motion is now absorbed into the integrable part. The remaining term, which is dependent on the colatitude, is “activated” only outside the equatorial plane. However, the effect of the colatitude-dependant terms on the new “unperturbed” motion is much smaller than the effect of the complete zonal potential on the Keplerian motion.

To generalize Theorem 1, the following condition for boundedness of relative motion, utilizing mean orbital elements, is written:²⁸

$$\dot{M}_0 + \dot{\omega}_0 + \dot{\Omega}_0 \cos i_0 = \dot{M}_1 + \dot{\omega}_1 + \dot{\Omega}_1 \cos i_1, \quad (84)$$

where M_k , ω_k , Ω_k , and i_k are the mean anomaly, argument of periapsis, right ascension of the ascending node and the inclination respectively, $k = 0, 1$. For equatorial orbits, condition (84) becomes:

$$\dot{M}_0 + \dot{\omega}_0 = \dot{M}_1 + \dot{\omega}_1. \quad (85)$$

Equation (85) is equivalent to:

$$\frac{2\pi}{T_0^{Kep}} + \dot{\omega}_0 = \frac{2\pi}{T_1^{Kep}} + \dot{\omega}_1, \quad (86)$$

where $T_{0,1}^{Kep}$ represent the periods of the osculating Keplerian nominal periods. In the case at hand, the nominal orbit is non-Keplerian. In the equatorial plane, condition (86) has the exact form:

$$T_0 = T_1; \varphi_0 = \varphi_1. \quad (87)$$

Note that (86) – even in the equatorial plane – involves an averaged osculating orbit, while (87) offers *exact analytical necessary and sufficient conditions* for periodicity.

The above considerations imply that the same periodicity conditions as in (87) should be applied when the absolute orbits are inclined. Moreover, from the secular and long-periodic variations of the mean orbital elements up to $\mathcal{O}(J_2^2)$, it may be deduced based on (84) that the same conditions (87) will remain valid when:

$$\Delta i \triangleq i_1 - i_0 \sim \mathcal{O}(J_2^2), \quad \Delta(M + \omega) \triangleq (M_1 - M_0) + (\omega_1 - \omega_0) \sim \mathcal{O}(J_2). \quad (88)$$

The difference $\Delta\Omega \triangleq \Omega_1 - \Omega_0$ may be chosen at will according to the mission specification, as it does not affect the boundedness of the relative motion.

ILLUSTRATIVE EXAMPLES

Numerical simulations were performed by generating the initial conditions as explained above, and then using them to initialize the High Precision Orbit Propagator of the STK[®] program. The orbits were propagated for 1 year. In each simulation, the relative distance between the satellites was computed for J_2 only and for the entire zonal potential up to order J_{21} .

Example 1 (periodic equatorial motion) *Determine the initial conditions of the deputy required to generate an equatorial periodic relative orbit in the presence of J_2 . The initial conditions of the chief, in terms of osculating elements, are: $a_0(0) = 8000$ km, $e_0(0) = 0.002$, $i_0(0) = 0^\circ$, $l_0(0) = 0^\circ$.*

The initial conditions of the chief in the inertial frame are:

$$\mathbf{r}_0^0 = \begin{bmatrix} 7984 \\ 0 \\ 0 \end{bmatrix} \text{ km}; \quad \mathbf{v}_0^0 = \begin{bmatrix} 0 \\ 7.07282 \\ 0 \end{bmatrix} \text{ km/sec.} \quad (89)$$

The values $r_{\min}^{(0)}$, $r_{\max}^{(0)}$, T_0 , φ_0 are determined to be:

$$r_{\min}^{(0)} = 7984 \text{ km}, \quad r_{\max}^{(0)} = 7999.4274 \text{ km}, \quad T_0 = 7113.6917 \text{ sec}, \quad \varphi_0 = 3.14484 \text{ rad.} \quad (90)$$

The commensurability conditions provide the following values for $r_{\min}^{(1)}$, $r_{\max}^{(1)}$:

$$r_{\min}^{(1)} = 7983.006 \text{ km}, \quad r_{\max}^{(1)} = 8000.42145 \text{ km.} \quad (91)$$

The values:

$$\varpi_1(0) = 0.001 \text{ rad}; \quad \nu_1(0) = -0.001 \text{ rad} \quad (92)$$

were chosen to initialize the deputy orbit. The following initial inertial position and velocity vectors for the deputy are obtained:

$$\mathbf{r}_1^0 = \begin{bmatrix} 7983.002 \\ 7.983 \\ 0 \end{bmatrix} \text{ km}; \quad \mathbf{v}_1^0 = \begin{bmatrix} -0.00708 \\ 7.0737 \\ 0 \end{bmatrix} \text{ km/sec.} \quad (93)$$

These values correspond to the following initial osculating orbital elements for the deputy: $a_1(0) = 7999.9894$ km, $e_1(0) = 0.00212$; $i_1(0) = 0^\circ$; $l_1(0) = 0^\circ$. The simulation results are shown in Fig. 3, which depicts the periodic relative orbit obtained.

Example 2 (inclined orbits, no initial differential inclination) *Determine the initial conditions of the deputy required to generate an inclined long-term bounded relative orbit in the presence of J_2 . The initial conditions of the chief, in terms of osculating elements, are set as follows: $a_0(0) = 8000$ km, $e_0(0) = 0.1$, $i_0(0) = 60^\circ$, $\omega_0(0) = 0^\circ$, $\Omega_0(0) = 0^\circ$, $\nu_0(0) = 0^\circ$.*

The initial conditions of the chief in the inertial frame are:

$$\mathbf{r}_0^0 = \begin{bmatrix} 7200 \\ 0 \\ 0 \end{bmatrix} \text{ km}; \quad \mathbf{v}_0^0 = \begin{bmatrix} 0 \\ 3.9018 \\ 6.7581 \end{bmatrix} \text{ km/sec.} \quad (94)$$

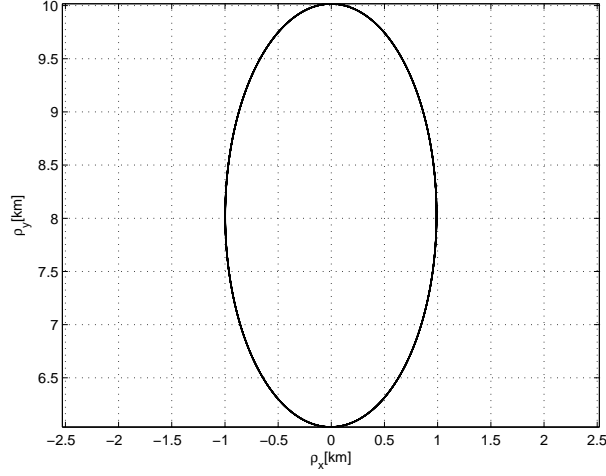


Figure 3. Relative orbit obtained in Example 1, J_2 perturbations only. As the theory predicts, a simple periodic orbit is obtained.

The values $r_{\min}^{(0)}$, $r_{\max}^{(0)}$, T_0 , φ_0 are determined using the chief's energy and initial angular momentum magnitude similarly to Example 1:

$$r_{\min}^0 = 7200 \text{ km}; r_{\max}^0 = 8779.353 \text{ km}, T_0 = 7111.0106 \text{ sec}, \varphi_0 = 3.144908 \text{ rad.} \quad (95)$$

The following values for $r_{\min}^{(1)}$, $r_{\max}^{(1)}$ are found:

$$r_{\min}^1 = 7199.902 \text{ km}; r_{\max}^1 = 8779.451 \text{ km.} \quad (96)$$

Based on the developments made previously in this paper, the following initial osculating orbital elements for the deputy are obtained: $a_1(0) = 7999.6868 \text{ km}$, $e_1(0) = 0.099$; $i_1(0) = 60^\circ$, $\omega_1(0) = 0.58^\circ$, $\Omega_1(0) = 0.50^\circ$, $\nu_1(0) = 359.01^\circ$. The values of $\omega_1(0)$, $\nu_1(0)$ were determined so as to satisfy the boundedness conditions (84). In terms of Cartesian inertial initial conditions,

$$\mathbf{r}_1^0 = \begin{bmatrix} 7200 \\ 0.0376 \\ -0.0435 \end{bmatrix} \text{ km}; \mathbf{v}_1^0 = \begin{bmatrix} 0.0082 \\ 3.9019 \\ 6.758 \end{bmatrix} \text{ km/sec.} \quad (97)$$

The results are illustrated in Fig. 4 for J_2 only and in Fig. 5 when all the harmonics up to J_{21} are taken into consideration. These figures, depicting the relative distance between the chief and deputy, clearly show that a bounded relative motion is obtained for a one-year period, with relative distances ranging from about 10 km to 70 km.

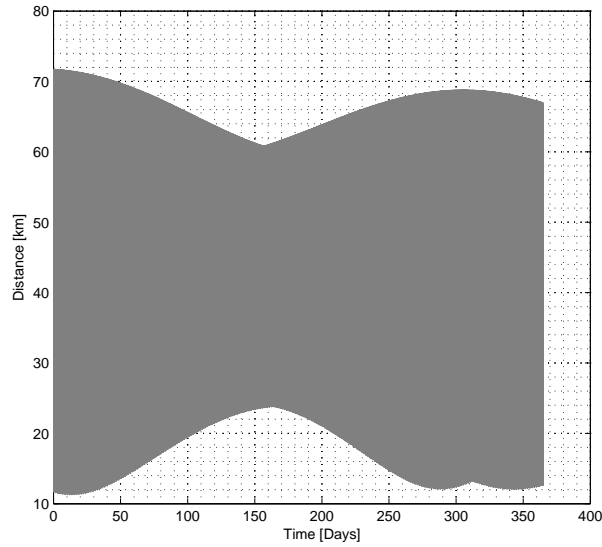


Figure 4. Relative distance between the chief and deputy satellites in Example 2, J_2 perturbations only. The relative distance remains bounded between about 10 km and 70 km for a year.

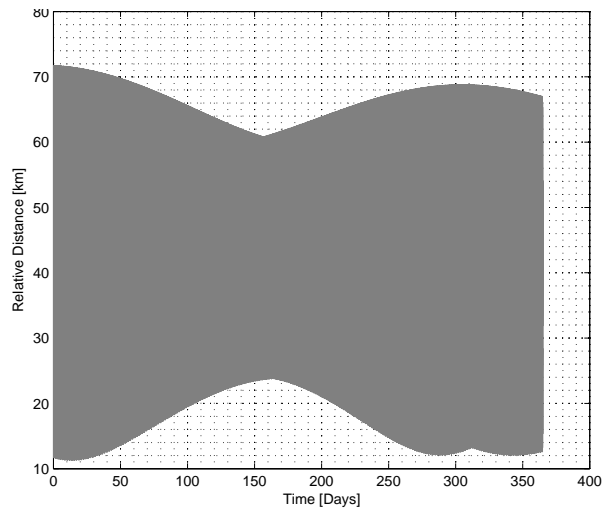


Figure 5. Relative distance between the chief and deputy satellites in Example 2, J_2 up to J_{21} perturbations. Similarly to Fig. 4, the relative distance remains bounded between about 10 km and 70 km for a year.

CONCLUSIONS

A different approach was proposed for the study of satellite relative motion in an axially-symmetric gravitational field. Instead of using the Keplerian motion as the generating nominal orbit for the absolute motion, another “unperturbed” orbit was proposed instead: An equatorial orbit about an oblate planet. Based on the superintegrability of such motion, closed-form solutions for the relative motion were derived in the situation where the planes of motion of both satellites lie in the equatorial plane. Analytic conditions for the periodicity of the relative motion in a generic central force field were presented, which were further used to determine the necessary and sufficient conditions for periodic relative motion. The necessary and sufficient periodicity conditions hold for any central motion, and can therefore be applied for equatorial motion under arbitrarily high-order even zonal harmonics. When applied to the case of the J_2 -perturbed motion in inclined orbits, these conditions are sufficient for the boundedness of relative motion. Furthermore, numerical simulations have shown that the derived conditions guarantee bounded (quasiperiodic) motion when high-order zonal harmonics (up to J_{21} in the numerical simulations) were present.

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