SIAM J. OPTIM.
Vol. 19, No. 2, pp. 786–807
© 2008 Society for Industrial and Applied Mathematics

ON THE BEHAVIOR OF SUBGRADIENT PROJECTIONS METHODS FOR CONVEX FEASIBILITY PROBLEMS IN EUCLIDEAN SPACES*

DAN BUTNARIU†, YAIR CENSOR†, PINI GURFIL‡, AND ETHAN HADAR§

Abstract. We study some methods of subgradient projections for solving a convex feasibility problem with general (not necessarily hyperplanes or half-spaces) convex sets in the inconsistent case and propose a strategy that controls the relaxation parameters in a specific self-adapting manner. This strategy leaves enough user flexibility but gives a mathematical guarantee for the algorithm's behavior in the inconsistent case. We present the numerical results of computational experiments that illustrate the computational advantage of the new method.

Key words. convex feasibility problems, projection method, computational algorithms

AMS subject classifications. 49M37, 90C25

DOI. 10.1137/070689127

1. Introduction.

In this paper we consider, in an Euclidean space framework, the method of simultaneous subgradient projections for solving a convex feasibility problem with general (not necessarily linear) convex sets in the consistent and inconsistent cases. To cope with this situation, we propose two algorithmic developments. One uses *steering parameters* instead of *relaxation parameters* in the simultaneous subgradient projection method, and the other is a strategy that controls the relaxation parameters in a specific self-adapting manner that leaves enough user flexibility while yielding some mathematical guarantees for the algorithm's behavior in the inconsistent case. For the algorithm that uses steering parameters there is currently no mathematical theory. We present the numerical results of computational experiments that show the computational advantage of the mathematically-founded algorithm implementing our specific relaxation strategy. In the remainder of this section we elaborate upon the meaning of the above-made statements.

Given m closed convex subsets $Q_1, Q_2, \ldots, Q_m \subseteq R^n$ of the n-dimensional Euclidean space, expressed as

$$Q_i = \{x \in R^n \mid f_i(x) \leq 0\}, \tag{1.1}$$

where $f_i : R^n \to R$ is a convex function, the *convex feasibility problem* (CFP) is

$$\text{find a point } x^* \in Q := \cap_{i=1}^m Q_i. \tag{1.2}$$

*Received by the editors April 23, 2007; accepted for publication (in revised form) March 6, 2008; published electronically July 3, 2008. This work was supported by grant 2003275 of the United States–Israel Binational Science Foundation, by National Institutes of Health grant HL70472, by The Technion–University of Haifa Joint Research Fund, and by grant 522/04 of the Israel Science Foundation at the Center for Computational Mathematics and Scientific Computation in the University of Haifa.
http://www.siam.org/journals/siopt/19-2/68912.html
†Department of Mathematics, University of Haifa, Mt. Carmel, Haifa 31905, Israel (dbutnaru@math.haifa.ac.il, yair@math.haifa.ac.il).
‡Faculty of Aerospace Engineering, Technion–Israel Institute of Technology, Technion City, Haifa 32000, Israel (pgurfil@aerodyne.technion.ac.il).
§CA Labs, Computer Associates International, Inc., Yokneam 20692, Israel (ethan.hadar@ca.com).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

As is well known, if the sets are given in any other form, then they can be represented in the form (1.1) by choosing, for f_i, the squared Euclidean distance to the set. Thus, it is required to solve the system of convex inequalities

$$f_i(x) \leq 0, \quad i = 1, 2, \ldots, m. \tag{1.3}$$

A fundamental question is how to approach the CFP in the inconsistent case when $Q = \cap_{i=1}^{m} Q_i = \emptyset$. Logically, algorithms designed to solve the CFP by finding a point $x^* \in Q$ are bound to fail and should, therefore, not be employed. But this is not always the case. Projection methods that are commonly used for the CFP, particularly in some very large real-world applications (see details below), are applied to CFPs without prior knowledge whether or not the problem is consistent. In such circumstances it is imperative to know how would a method, that is originally known to converge for a consistent CFP, behave if consistency is not guaranteed.

We address this question for a particular type of projection methods. In general, sequential projection methods exhibit *cyclic convergence* in the inconsistent case. This means that the whole sequence of iterates does not converge, but it breaks up into m convergent subsequences (see Gubin, Polyak, and Raik [25, Theorem 2] and Bauschke, Borwein and Lewis [3]). In contrast, simultaneous projection methods generally converge, even in the inconsistent case, to a minimizer of a proximity function that "measures" the weighted sum of squared distances to all sets of the CFP provided such a minimizer exists (see Iusem and De Pierro [28] for a local convergence proof and Combettes [17] for a global one).

Therefore, there is an advantage in using simultaneous projection methods from the point of view of convergence. Additional advantages are that (i) they are inherently parallel already at the mathematical formulation level due to the simultaneous nature, and (ii) they allow the user to assign weights (of importance) to the sets of the CFP. However, a severe limitation, common to sequential as well as simultaneous projection methods, is the need to solve an inner-loop distance-minimization step for the calculation of the orthogonal projection onto each individual set of the CFP. This need is alleviated only for convex sets that are simple to project onto, such as hyperplanes or half-spaces.

A useful path to circumvent this limitation is to use subgradient projections that rely on the calculation of subgradients at the current (available) iteration points; see Censor and Lent [12] or [13, section 5.3]. Iusem and Moledo [32] studied the simultaneous projection method with subgradient projections but only for consistent CFPs. To the best of our knowledge, there does not exist a study of the simultaneous projection method with subgradient projections for the inconsistent case. Our present results are a contribution towards this goal.

The CFP is a fundamental problem in many areas of mathematics and the physical sciences; see, e.g., Combettes [16, 18] and references therein. It has been used to model significant real-world problems in image reconstruction from projections; see, e.g., Herman [26], in radiation therapy treatment planning; see Censor, Altschuler, and Powlis [11] and Censor [9], and in crystallography; see Marks, Sinkler and Landree [33], to name but a few, and has been used under additional names such as *set-theoretic estimation* or the *feasible set approach*. A common approach to such problems is to use projection algorithms; see, e.g., Bauschke and Borwein [2], which employ *orthogonal projections* (i.e., nearest-point mappings) onto the individual sets Q_i. The orthogonal projection $P_\Omega(z)$ of a point $z \in R^n$ onto a closed convex set $\Omega \subseteq R^n$ is defined by

$$P_\Omega(z) := \text{argmin}\{\| z - x \| \mid x \in \Omega\}, \tag{1.4}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

where, throughout this paper, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean norm and inner product, respectively, in R^n. Frequently a *relaxation parameter* is introduced so that

$$P_{\Omega,\lambda}(z) := (1-\lambda)z + \lambda P_\Omega(z) \tag{1.5}$$

is the *relaxed projection* of z onto Ω with relaxation λ. Many iterative projection algorithms for the CFP were developed; see subsection 1.1 below.

1.1. Projection methods: Advantages and earlier work. The reason why the CFP is looked at from the viewpoint of projection methods can be appreciated by the following brief comments, that we made in earlier publications, regarding projection methods in general. Projections onto sets are used in a variety of methods in optimization theory, but not every method that uses projections really belongs to the class of projection methods. *Projection methods* are iterative algorithms which use projections onto sets. They rely on the general principle that projections onto the given individual sets are easier to perform than projections onto other sets derived from the given individual sets (intersections, image sets under some transformation, etc.).

A projection algorithm reaches its goal, related to the whole family of sets, by performing projections onto the individual sets. Projection algorithms employ projections onto convex sets in various ways. They may use different kinds of projections and, sometimes, even use different types of projections within the same algorithm. They serve to solve a variety of problems, which are of either the feasibility or of the optimization types. They have different algorithmic structures, of which some are particularly suitable for parallel computing, and they demonstrate nice convergence properties and/or good initial behavior patterns.

Apart from theoretical interest, the main advantage of projection methods, which makes them successful in real-world applications, is computational. They commonly have the ability to handle huge-size problems that are beyond the ability of more sophisticated and currently available methods. This is so because the building blocks of a projection algorithm are the projections onto the given individual sets (which are easy to perform), and the algorithmic structure is either sequential or simultaneous (or in-between).

The field of projection methods is vast, and we mention here only a few recent works that can give the reader some good starting points. Such a list includes, among many others, the works of Crombez [20, 21], the connection with variational inequalities; see, e.g., Noor [34] and Yamada [38], which is motivated by real-world problems of signal processing, and the many contributions of Bauschke and Combettes; see, e.g., Bauschke, Combettes, and Kruk [4] and references therein. Bauschke and Borwein [2] and Censor and Zenios [13, Chapter 5] provide reviews of the field.

Systems of linear equations, linear inequalities, or convex inequalities are all encompassed by the CFP, which has broad applicability in many areas of mathematics and the physical and engineering sciences. These include, among others, optimization theory (see, e.g., Eremin [24], Censor and Lent [12], and Chinneck [14]), approximation theory (see, e.g., Deutsch [22] and references therein), image reconstruction from projections in computerized tomography (see, e.g., Herman [26, 27]), and control theory (see, e.g., Boyd et al. [5].)

Combettes [19] and Kiwiel [31] have studied the subgradient projection method for consistent CFPs. Their work presents more general algorithmic steps and is formulated in Hilbert space. Some work has already been done on detecting infeasibility with certain subgradient projection methods by Kiwiel [29, 30]. However, our ap-

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

proach differs from the latter in that it aims at a subgradient projection method that "will work" regardless of the feasibility of the underlying CFP and which does not require the user to study in advance whether or not the CFP is consistent. Further questions arise such as that of combining our work, or the above quoted results, with Pierra's [36] product space formalism, as extended to handle inconsistent situations by Combettes [17]. These questions are currently under investigation.

2. Simultaneous subgradient projections with steering parameters. Subgradient projections have been incorporated in iterative algorithms for the solution of CFPs. The cyclic subgradient projections (CSP) method for the CFP was given by Censor and Lent [12] as follows.

ALGORITHM 2.1 (The CSP method).

Initialization: *$x^0 \in R^n$ is arbitrary.*

Iterative step: *Given x^k, calculate the next iterate x^{k+1} by*

$$x^{k+1} = \begin{cases} x^k - \alpha_k \dfrac{f_{i(k)}(x^k)}{\parallel t^k \parallel^2} t^k & \text{if } f_{i(k)}(x^k) > 0, \\ x^k & \text{if } f_{i(k)}(x^k) \leq 0, \end{cases} \tag{2.1}$$

where $t^k \in \partial f_{i(k)}(x^k)$ is a subgradient of $f_{i(k)}$ at the point x^k, and the relaxation parameters $\{\alpha_k\}_{k=0}^{\infty}$ are confined to an interval $\epsilon_1 \leq \alpha_k \leq 2 - \epsilon_2$, for all $k \geq 0$, with some arbitrarily small $\epsilon_1, \epsilon_2 > 0$.

Control: *Denoting $I := \{1, 2, \ldots, m\}$, the sequence $\{i(k)\}_{k=0}^{\infty}$ is an almost cyclic control sequence on I. This means (see, e.g.,* [13, *Definition* 5.1.1]*) that $i(k) \in I$ for all $k \geq 0$, and there exists an integer $C \geq m$ such that, for all $k \geq 0$, $I \subseteq \{i(k+1), i(k+2), \ldots, i(k+C)\}$.*

Observe that if $t^k = 0$, then $f_{i(k)}$ takes its minimal value at x^k, implying, by the nonemptiness of Q, that $f_{i(k)}(x^k) \leq 0$ so that $x^{k+1} = x^k$. The relations of the CSP method to other iterative methods for solving the convex feasibility problem and to the relaxation method for solving linear inequalities can be found, e.g., in [13, Chapter 5]; see also, Bauschke and Borwein [2, section 7]. Since sequential projection methods for CFPs commonly have fully simultaneous counterparts, the simultaneous subgradient projections (SSP) method of Dos Santos [23] and Iusem and Moledo [32] is a natural algorithmic development.

ALGORITHM 2.2 (The SSP method).

Initialization: *$x^0 \in R^n$ is arbitrary.*

Iterative step: (i) *Given x^k, calculate, for all $i \in I = \{1, 2, \ldots, m\}$, intermediate iterates $y^{k+1,i}$ by*

$$y^{k+1,i} = \begin{cases} x^k - \alpha_k \dfrac{f_i(x^k)}{\parallel t^k \parallel^2} t^k & \text{if } f_i(x^k) > 0, \\ x^k & \text{if } f_i(x^k) \leq 0, \end{cases} \tag{2.2}$$

where $t^k \in \partial f_i(x^k)$ is a subgradient of f_i at the point x^k, and the relaxation parameters $\{\alpha_k\}_{k=0}^{\infty}$ are confined to an interval $\epsilon_1 \leq \alpha_k \leq 2 - \epsilon_2$, for all $k \geq 0$, with some arbitrarily small $\epsilon_1, \epsilon_2 > 0$.

(ii) *Calculate the next iterate x^{k+1} by*

$$x^{k+1} = \sum_{i=1}^{m} w_i y^{k+1,i}, \tag{2.3}$$

where w_i are fixed, user-chosen, positive weights with $\sum_{i=1}^{m} w_i = 1$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

The convergence analysis for this algorithm is currently available only for consistent ($Q \neq \emptyset$) CFPs; see [23, 32]. In our experimental work, reported in what follows, we applied Algorithm 2.2 to CFPs without knowing whether or not they are consistent. Convergence is diagnosed by performing the plots of a *proximity function* that measures, in some manner, the infeasibility of the system. We used the weighted proximity function of the form

$$p(x) := (1/2) \sum_{i=1}^{m} w_i \parallel P_i(x) - x \parallel^2, \tag{2.4}$$

where $P_i(x)$ is the orthogonal projection of the point x onto Q_i. To combat instabilities in those plots that appeared occasionally in our experiments, we used *steering parameters* σ_k instead of the relaxation parameters α_k in Algorithm 2.2. To this end we need the following definition.

DEFINITION 2.3. *A sequence* $\{\sigma_k\}_{k=0}^{\infty}$ *of real numbers* $0 \leq \sigma_k < 1$ *is called a* steering sequence *if it satisfies the following conditions:*

$$\lim_{k \to \infty} \sigma_k = 0, \tag{2.5}$$

$$\sum_{k=0}^{\infty} \sigma_k = +\infty, \tag{2.6}$$

$$\sum_{k=0}^{\infty} \mid \sigma_k - \sigma_{k+m} \mid < +\infty. \tag{2.7}$$

A historical and technical discussion of these conditions can be found in [1]. The sequential and simultaneous Halpern–Lions–Wittmann–Bauschke algorithms discussed in Censor [10] employ the parameters of a steering sequence to "force" (steer) the iterates towards the solution of the best approximation problem. This steering feature of the steering parameters has a profound effect on the behavior of any sequence of iterates $\{x^k\}_{k=0}^{\infty}$. We return to this point in section 6.

ALGORITHM 2.4 (The SSP method with steering).

Initialization: $x^0 \in R^n$ *is arbitrary.*

Iterative step: (i) *Given* x^k, *calculate, for all* $i \in I = \{1, 2, \ldots, m\}$, *intermediate iterates* $y^{k+1,i}$ *by*

$$y^{k+1,i} = \begin{cases} x^k - \sigma_k \dfrac{f_i(x^k)}{\parallel t^k \parallel^2} t^k & \text{if } f_i(x^k) > 0, \\ x^k & \text{if } f_i(x^k) \leq 0, \end{cases} \tag{2.8}$$

where $t^k \in \partial f_i(x^k)$ *is a subgradient of* f_i *at the point* x^k, *and* $\{\sigma_k\}_{k=0}^{\infty}$ *is a sequence of steering parameters.*

(ii) *Calculate the next iterate* x^{k+1} *by*

$$x^{k+1} = \sum_{i=1}^{m} w_i y^{k+1,i}, \tag{2.9}$$

where w_i *are fixed, user-chosen, positive weights with* $\sum_{i=1}^{m} w_i = 1$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

3. Subgradient projections with strategical relaxation: Preliminaries. Considering the CFP (1.2), the *envelope* of the family of functions $\{f_i\}_{i=1}^m$ is the function

$$f(x) := \max\{f_i(x) \mid i = 1, 2, \ldots, m\}, \tag{3.1}$$

which is also convex. Clearly, the consistent CFP is equivalent to finding a point in

$$Q = \cap_{i=1}^m Q_i = \{x \in R^n \mid f(x) \leq 0\}. \tag{3.2}$$

The subgradient projections algorithmic scheme that we propose here employs a strategy for controlling the relaxation parameters in a specific manner, leaving enough user flexibility while giving some mathematical guarantees for the algorithm's behavior in the inconsistent case. It is described as follows.

ALGORITHM 3.1.

Initialization: *Let M be a positive real number, and let $x^0 \in R^n$ be any initial point.*

Iterative step: *Given the current iterate x^k, set*

$$I(x^k) := \{i \mid 1 \leq i \leq m \quad and \quad f_i(x^k) = f(x^k)\}, \tag{3.3}$$

and choose a nonnegative vector $w^k = (w_1^k, w_2^k, \ldots, w_m^k) \in R^m$ such that

$$\sum_{i=1}^m w_i^k = 1 \quad and \quad w_i^k = 0 \quad if \quad i \notin I(x^k). \tag{3.4}$$

Let λ_k be any nonnegative real number such that

$$\max\left(0,\ f(x^k)\right) \leq \lambda_k M^2 \leq 2 \max\left(0,\ f(x^k)\right) \tag{3.5}$$

and calculate

$$x^{k+1} = x^k - \lambda_k \sum_{i \in I(x^k)} w_i^k \xi_i^k, \tag{3.6}$$

where, for each $i \in I(x^k)$, we take a subgradient $\xi_i^k \in \partial f_i(x^k)$.

It is interesting to note that any sequence $\{x^k\}_{k=0}^\infty$ generated by this algorithm is well defined, no matter how x^0 and M are chosen. Similarly to other algorithms described above, Algorithm 3.1 requires computing the subgradients of convex functions. In case a function is differentiable, this reduces to gradient calculations. Otherwise, one can use the subgradient computing procedure presented in Butnariu and Resmerita [8].

The procedure described above was previously studied in Butnariu and Mehrez [7]. The main result there shows that the procedure converges to a solution of the CFP under two conditions: (i) that the solution set Q has a nonempty interior, and (ii) that the envelope f is uniformly Lipschitz on R^n, that is, there exists a positive real number L such that

$$|f(x) - f(y)| \leq L \|x - y\| \quad \text{for all} \quad x, y \in R^n. \tag{3.7}$$

Both conditions (i) and (ii) are restrictive, and it is difficult to verify their validity in practical applications. In the following we show that this method converges to the

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

solutions of consistent CFPs under less demanding conditions. In fact, we show that if the solution set Q of the given CFP has a nonempty interior, then the convergence of Algorithm 3.1 to a point in Q is ensured even if the function f is not uniformly Lipschitz on R^n (i.e., even if f does not satisfy condition (ii) above). However, verifying whether $\operatorname{int} Q \neq \emptyset$ prior to solving a CFP may be difficult or even impossible. Therefore, it is desirable to have alternative conditions, which may be easier to verify in practice, that can ensure convergence of our algorithm to solutions of the CFP, provided that such solutions exist. This is why we prove the convergence of Algorithm 3.1 to the solutions of consistent CFPs whenever the envelope f of the functions f_i involved in the given CFP is strictly convex. The strict convexity of the envelope function f associated with a consistent CFP implies that either the solution set Q of the CFP is a singleton, in which case $\operatorname{int} Q = \emptyset$, or that Q contains (at least) two different solutions of the CFP implying that $\operatorname{int} Q \neq \emptyset$. The verification of whether Q is a singleton or not is as difficult as deciding whether $\operatorname{int} Q \neq \emptyset$. By contrast, since f is strictly convex whenever each f_i is strictly convex, the verification of the strict convexity of f may be relatively easily done in some situations of practical interest, such as when each f_i is a quadratic convex function. In the latter case, strict convexity of f_i amounts to the positive definiteness of the matrix of its purely quadratic part.

It is interesting to note in this context that, when the envelope f of the CFP is not strictly convex, one may consider a "regularized" CFP in which each f_i, which is not strictly convex, is replaced by

$$\overline{f_i}(x) := f_i(x) + \alpha \|x\|^2 \tag{3.8}$$

for some positive real number α. Clearly, all $\overline{f_i}$ are strictly convex, and thus so is the envelope $\overline{f}$ of the regularized problem. Therefore, if the regularized problem has solutions, then our Algorithm 3.1 will produce the approximations of such solutions. Moreover, any solution of the regularized problem is a solution of the original problem, and thus by solving the regularized problem, we implicitly solve the original problem. The difficult part of this approach is that, even if the original CFP is consistent, then the regularized version of it may be inconsistent for all, or for some, values $\alpha > 0$. How to decide whether an $\alpha > 0$ exists such that the corresponding regularized CFP is consistent, and how to compute such an α (if any) are questions whose answers we do not know.

4. Subgradient projections with strategical relaxation: Convergence analysis.

In order to discuss the convergence behavior of the subgradient projections method with strategical relaxation, recall that convex functions defined on the whole space R^n are continuous and, consequently, are bounded on bounded sets in R^n. Therefore, the application of Butnariu and Iusem [6, Proposition 1.1.11] or Bauschke and Borwein [2, Proposition 7.8] to the convex function f shows that it is Lipschitz on bounded subsets of R^n, i.e., for any nonempty bounded subset $S \subseteq R^n$, there exists a positive real number $L(S)$, called *a Lipschitz constant of f over the set S*, such that

$$|f(x) - f(y)| \leq L(S)\,\|x - y\| \;\text{ for all }\; x, y \in S. \tag{4.1}$$

Our next result is a convergence theorem for Algorithm 3.1 when applied to a consistent CFP. It was noted in the previous section that Algorithm 3.1 is well defined regardless of how the initial point x^0 or the positive constant M involved in the algorithm are chosen. However, this is no guarantee that a sequence $\{x_k\}_{k=0}^{\infty}$ generated by Algorithm 3.1 for the random choices of x^0 and M will converge to the solutions

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

of the CFP, even if such solutions exist. The theorem below shows a way of choosing x^0 and M, which ensures that, under some additional conditions for the problem data, the sequence $\{x_k\}_{k=0}^{\infty}$ generated by Algorithm 3.1 will necessarily approximate a solution of the CFP (provided that solutions exist). As shown in section 5 below, determining x^0 and M as required in the next theorem can be quite easily done for practically significant classes of CFPs. Also, as shown in section 6, determining x^0 and M in this manner enhances the self-adaptability of the procedure to the problem data and makes Algorithm 3.1 produce the approximations of the solutions of the CFP which, in many cases, are more accurate than those produced by other CFP solving algorithms. We denote with $B(x, r)$ the ball centered at x with radius r.

THEOREM 4.1. *If a positive number M and an initial point x^0 in Algorithm* 3.1 *are chosen so that $M \geq L(B(x^0, r))$ for some positive real number r satisfying the condition*

$$B(x^0, r/2) \cap Q \neq \emptyset \tag{4.2}$$

and if at least one of the following conditions holds:

(i) $B(x^0, r/2) \cap \operatorname{int} Q \neq \emptyset$,

(ii) *the function f is strictly convex,*

then any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm 3.1*, converges to an element of Q.*

We present the proof of Theorem 4.1 as a sequence of lemmas. To do so, note that, if $\{x^k\}_{k=0}^{\infty}$ is generated by Algorithm 3.1, then for each integer $k \geq 0$, we have

$$x^{k+1} = x^k - \lambda_k \nu^k, \tag{4.3}$$

where

$$\nu^k := \sum_{i \in I(x^k)} w_i^k \xi_i^k \in \operatorname{conv} \cup_{i \in I(x^k)} \partial f_i(x^k). \tag{4.4}$$

Using (4.3), for any $z \in R^n$, we have

$$\left\|x^{k+1} - z\right\|^2 = \left\|x^k - z\right\|^2 + \lambda_k \left(\lambda_k \left\|\nu^k\right\|^2 - 2\left\langle \nu^k, x^k - z\right\rangle\right). \tag{4.5}$$

By Clarke [15, Proposition 2.3.12] we deduce that

$$\partial f(x^k) = \operatorname{conv} \cup_{i \in I(x^k)} \partial f_i(x^k), \tag{4.6}$$

and this implies that $\nu^k \in \partial f(x^k)$ because of (4.4). Therefore,

$$\left\langle \nu^k, z - x^k\right\rangle \leq f'_+(x^k; z - x^k), \tag{4.7}$$

where $f'_+(u; v)$ denotes the right-sided directional derivative at u in the direction v. Now suppose that M, r, and x^0 are chosen according to the requirements of Theorem 4.1, that is,

$$r > 0, M \geq L(B(x^0, r)) \text{ and } B(x^0, r/2) \cap Q \neq \emptyset. \tag{4.8}$$

Next we prove the following basic fact.

LEMMA 4.2. *If* (4.8) *is satisfied and if $z \in B(x^0, r/2) \cap Q$, then, for all $k \geq 0$, we have, for any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm* 3.1,

$$x^{k+1} \in B(x^0, r) \text{ and } \left\|x^{k+1} - z\right\| \leq \left\|x^k - z\right\| \leq r/2. \tag{4.9}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Proof. We first show that, if for some integer $k \geq 0$,

$$x^k \in B(x^0, r) \text{ and } \left\|x^k - z\right\| \leq r/2, \tag{4.10}$$

then (4.9) holds. If $\lambda_k = 0$ or $\nu^k = 0$, then, by (4.3), we have $x^{k+1} = x^k$, which combined with (4.10), implies (4.9). Assume now that $\lambda_k \neq 0$ and $\nu^k \neq 0$. Since, by (4.10), $x^k \in B(x^0, r)$ by (4.8) and by [15, Proposition 2.1.2(a)], we deduce that

$$M \geq L(B(x^0, r)) \geq \left\|\nu^k\right\|. \tag{4.11}$$

According to (3.5), we also have $f(x^k) > 0$ (otherwise $\lambda_k = 0$). Since $f(z) \leq 0$ we obtain from the subgradient inequality

$$\left\langle \nu^k, x^k - z\right\rangle \geq f(x^k) - f(z) \geq f(x^k) > 0. \tag{4.12}$$

This and (4.11) imply that

$$2\left\langle \nu^k, x^k - z\right\rangle \geq 2f(x^k) \geq \lambda_k M^2 \geq \lambda_k \left\|\nu^k\right\|^2 \tag{4.13}$$

showing that the quantity inside the parentheses in (4.5) is nonpositive. Thus, we deduce that

$$\left\|x^{k+1} - z\right\| \leq \left\|x^k - z\right\| \leq r/2 \tag{4.14}$$

in this case too. This proves that if (4.10) is true for all $k \geq 0$, then so is (4.9). Now, we prove by induction that (4.10) is true for all $k \geq 0$. If $k = 0$, then (4.10) obviously holds. Suppose that (4.10) is satisfied for some $k = p$. As shown above, this implies that condition (4.9) is satisfied for $k = p$, and thus we have that

$$x^{p+1} \in B(x^0, r) \text{ and } \left\|x^{p+1} - z\right\| \leq r/2. \tag{4.15}$$

Hence, condition (4.10) also holds for $k = p + 1$. Consequently, condition (4.9) holds for $k = p + 1$, and this completes the proof. ∎

Observe that, according to Lemma 4.2, if $\{x^k\}_{k=0}^{\infty}$ is a sequence generated by Algorithm 3.1 and if the conditions (4.8) are satisfied, then there exists $z \in B(x^0, r/2) \cap Q$ and for any such z the sequence $\{\|x^k - z\|\}_{k=0}^{\infty}$ is nonincreasing and bounded from below and therefore convergent. Since the sequence $\{\|x^k - z\|\}_{k=0}^{\infty}$ is convergent, it is also bounded, and consequently, the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded too. This shows that the next result applies to any sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 3.1 under the assumptions of Theorem 4.1.

LEMMA 4.3. *If $\{x^k\}_{k=0}^{\infty}$ is a bounded sequence generated by Algorithm* 3.1, *then the sequence $\{x^k\}_{k=0}^{\infty}$ has accumulation points, and, for each accumulation point x^* of $\{x^k\}_{k=0}^{\infty}$, there exists a sequence of natural numbers $\{k_s\}_{s=0}^{\infty}$ such that the following limits exist:*

$$x^* = \lim_{s\to\infty} x^{k_s}, \ \lambda_* = \lim_{s\to\infty} \lambda_{k_s}, \tag{4.16}$$

$$\xi_i^* = \lim_{s\to\infty} \xi_i^{k_s}, \ w_i^* = \lim_{s\to\infty} w_i^{k_s} \text{ for all } i = 1, 2, \ldots, m, \tag{4.17}$$

$$\nu^* = \lim_{s\to\infty} \nu^{k_s}, \tag{4.18}$$

and we have

$$w^* := (w_1^*, w_2^*, \ldots, w_m^*) \in R_+^m \text{ and } \sum_{i\in I(x^*)} w_i^* = 1 \tag{4.19}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

and

$$\nu^* = \sum_{i \in I(x^*)} w_i^* \xi_i^* \in \partial f(x^*). \tag{4.20}$$

Moreover, if $\lambda_* = 0$, *then* x^* *is a solution of the CFP.*

Proof. The sequence $\{x^k\}_{k=0}^\infty$ is bounded, and thus has accumulation points. Let x^* be an accumulation point of $\{x^k\}_{k=0}^\infty$, and let $\{x^{p_s}\}_{s=0}^\infty$ be a convergent subsequence of $\{x^k\}_{k=0}^\infty$ such that $x^* = \lim_{s\to\infty} x^{p_s}$. The function f is continuous (since it is real-valued and convex on R^n); hence, it is bounded on bounded subsets of R^n. Therefore, the sequence $\{f(x^{p_s})\}_{s=0}^\infty$ converges to $f(x^*)$, and the sequence $\{f(x^k)\}_{k=0}^\infty$ is bounded. By (3.5), the boundedness of $\{f(x^k)\}_{k=0}^\infty$ implies that the sequence $\{\lambda_k\}_{k=0}^\infty$ is bounded. Since, for every $i = 1, 2, \ldots, m$, the operator $\partial f_i : R^n \to 2^{R^n}$ is monotone, it is locally bounded (cf. Pascali and Sburlan [35, Theorem on p. 104]).

Consequently, there exists a neighborhood U of x^* on which all ∂f_i, $i = 1, 2, \ldots, m$ are bounded. Clearly, since $x^* = \lim_{s\to\infty} x^{p_s}$, the neighborhood U contains all but finitely many terms of the sequence $\{x^{p_s}\}_{s=0}^\infty$. This implies that the sequences $\{\xi_i^{p_s}\}_{s=0}^\infty$ are uniformly bounded, and therefore the sequence $\{\nu^{p_s}\}_{s=0}^\infty$ is bounded too.

Therefore, there exists a subsequence $\{k_s\}_{s=0}^\infty$ of $\{p_s\}_{s=0}^\infty$ such that the limits in (4.16)–(4.18) exist. Obviously, the vector $w^* = (w_1^*, w_2^*, \ldots, w_m^*) \in R_+^m$, and according to [7, Lemma 1], we also have $\sum_{i\in I(x^*)} w_i^* = 1$. This and (4.4) imply that $\nu^* = \sum_{i\in I(x^*)} w_i^* \xi_i^*$.

Observe that, since $\nu^{k_s} \in \partial f(x^{k_s})$ for all $s \geq 0$ and since ∂f is a closed mapping (cf. Phelps [37, Proposition 2.5]), we have that $\nu^* \in \partial f(x^*)$. Now, if $\lambda_* = 0$, then according to (3.5) and the continuity of f, we deduce that

$$0 \leq \max\{0, f(x^*)\} = \lim_{s\to\infty} \max\{0, f(x^{k_s})\} \leq \lim_{s\to\infty} \lambda_{k_s} M^2 = \lambda_* M^2 = 0, \tag{4.21}$$

which implies that $f(x^*) \leq 0$, that is, $x^* \in Q$. ◻

LEMMA 4.4. *Let* $\{x^k\}_{k=0}^\infty$ *be a sequence generated by Algorithm* 3.1. *If* (4.8) *is satisfied and if at least one of the conditions* (i) *or* (ii) *of Theorem* 4.1 *holds, then the sequence* $\{x^k\}_{k=0}^\infty$ *has accumulation points, and any such point belongs to* Q.

Proof. As noted above, when (4.8) is satisfied, then the sequence $\{x^k\}_{k=0}^\infty$ is bounded, and hence it has accumulation points. Let x^* be such an accumulation point, and let $\{k_s\}_{s=0}^\infty$ be the sequence of natural numbers associated with x^* whose existence is guaranteed by Lemma 4.3. Since, for any $z \in C \cap B(x^0, r/2)$, the sequence $\{\|x^k - z\|\}_{k=0}^\infty$ is convergent (cf. Lemma 4.2), we deduce that

$$\begin{aligned} \|x^* - z\| &= \lim_{s\to\infty} \left\|x^{k_s} - z\right\| = \lim_{k\to\infty} \left\|x^k - z\right\| = \lim_{s\to\infty} \left\|x^{k_s+1} - z\right\| \\ &= \|x^* - \lambda_* \nu^* - z\| . \end{aligned} \tag{4.22}$$

This implies that

$$\|x^* - z\|^2 = \|x^* - z\|^2 + \lambda_* \left(\lambda_* \|\nu^*\|^2 - 2 \langle \nu^*, x^* - z\rangle\right). \tag{4.23}$$

If $\lambda_* = 0$, then $x^* \in Q$ by Lemma 4.3. Suppose that $\lambda_* > 0$. Then, by (4.23), we have

$$\lambda_* \|\nu^*\|^2 - 2 \langle \nu^*, x^* - z\rangle = 0 \tag{4.24}$$

for all $z \in C \cap B(x^0, r/2)$. We distinguish now between two possible cases.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Case I: Assume that condition (i) of Theorem 4.1 is satisfied. According to (4.24), the set $Q \cap B(x^0, r/2)$ is contained in the hyperplane

$$H := \left\{ x \in R^n \mid \langle \nu^*, x \rangle = (1/2) \left(2 \langle \nu^*, x^* \rangle - \lambda_* \|\nu^*\|^2 \right) \right\}. \tag{4.25}$$

By condition (i) of Theorem 4.1, it follows that $\operatorname{int}(Q \cap B(x^0, r/2)) \neq \emptyset$, and this is an open set contained in $\operatorname{int} H$. So, unless $\nu^* = 0$ (in which case $H = R^n$), we have reached a contradiction because $\operatorname{int} H = \emptyset$. Therefore, we must have $\nu^* = 0$. According to Lemma 4.3, we have $0 = \nu^* \in \partial f(x^*)$, which implies that x^* is a global minimizer of f. Consequently, for any $z \in Q$, we have $f(x^*) \leq f(z) \leq 0$, that is, $x^* \in Q$.

Case II: Assume that condition (ii) of Theorem 4.1 is satisfied. According to (4.24), we have

$$\lambda_* \|\nu^*\|^2 = 2 \langle \nu^*, x^* - z \rangle. \tag{4.26}$$

By (3.5), the definition of M, and [15, Proposition 2.1.2] we deduce that

$$2f(x^{k_s}) \geq \lambda_{k_s} M^2 \geq \lambda_{k_s} \left\|\nu^{k_s}\right\|^2 \tag{4.27}$$

for all integers $s \geq 0$. Letting $s \to \infty$ we get

$$2f(x^*) \geq \lambda_* M^2 \geq \lambda_* \|\nu^*\|^2 = 2 \langle \nu^*, x^* - z \rangle, \tag{4.28}$$

where the last equality follows from (4.26). Consequently, we have

$$f(x^*) \geq \langle \nu^*, x^* - z \rangle \quad \text{for all} \quad z \in Q \cap B(x^0, r/2). \tag{4.29}$$

The convexity of f implies that, for all $z \in Q \cap B(x^0, r/2)$,

$$-f(x^*) \leq \langle \nu^*, z - x^* \rangle \leq f(z) - f(x^*) \leq -f(x^*). \tag{4.30}$$

Therefore, we have that

$$-f(x^*) = \langle \nu^*, z - x^* \rangle = f(z) - f(x^*) \text{ for all } \quad z \in Q \cap B(x^0, r/2). \tag{4.31}$$

Thus $f(z) = 0$, for all $z \in Q \cap B(x^0, r/2)$. Hence, using again the convexity of f, we deduce that, for all $z \in Q \cap B(x^0, r/2)$,

$$f'_+(x^*; z - x^*) \leq f(z) - f(x^*) = -f(x^*) = \langle \nu^*, z - x^* \rangle \leq f'_+(x^*; z - x^*). \tag{4.32}$$

This implies that

$$f'_+(x^*; z - x^*) = \langle \nu^*, z - x^* \rangle = f(z) - f(x^*) \text{ for all } \quad z \in Q \cap B(x^0, r/2). \tag{4.33}$$

Since, by condition (ii) of Theorem 4.1, f is strictly convex, we also have (see [6, Proposition 1.1.4]) that

$$f'_+(x^*; z - x^*) < f(z) - f(x^*) \text{ for all } \quad z \in \left(Q \cap B(x^0, r/2)\right) \backslash \{x^*\}. \tag{4.34}$$

Hence, the equalities in (4.33) cannot hold unless $Q \cap B(x^0, r/2) = \{x^*\}$, and thus $x^* \in Q$. ◻

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

The previous lemmas show that if (4.8) holds and if one of the conditions (i) or (ii) of Theorem 4.1 is satisfied, then the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded, and all of its accumulation points are in Q. In fact, the results above say something more. Namely, in view of Lemma 4.2, they show that if (4.8) holds and if one of the conditions (i) or (ii) of Theorem 4.1 is satisfied, then all of the accumulation points x^* of $\{x^k\}_{k=0}^{\infty}$ are contained in $Q \cap B(x^0, r)$ because all x^k are in $B(x^0, r)$ by (4.9). In order to complete the proof of Theorem 4.1, it remains to show that the following result is true.

LEMMA 4.5. *Under the conditions of Theorem* 4.1 *any sequence* $\{x^k\}_{k=0}^{\infty}$, *generated by Algorithm* 3.1, *has at most one accumulation point.*

Proof. Observe that, under the conditions of Theorem 4.1, the conditions (4.8) are satisfied, and therefore the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded. Let x^* be an accumulation point of $\{x^k\}_{k=0}^{\infty}$. By Lemma 4.4 we deduce that $x^* \in Q$, i.e., $f(x^*) \leq 0$. Consequently, for any natural number k we have

$$\left\langle \nu^k, x^k - x^* \right\rangle \geq f(x^k) - f(x^*) \geq f(x^k).$$

Now, using this fact, a reasoning similar to that which proves (4.13) but made with x^* instead of z leads to

$$2 \left\langle \nu^k, x^k - x^* \right\rangle \geq \lambda_k \left\| \nu^k \right\|^2$$

for all natural numbers k. This and (4.5) combined imply that the sequence $\{\|x^k - x^*\|\}_{k=0}^{\infty}$ is nonincreasing and therefore convergent. Consequently, if $\{x^{k_p}\}_{p=0}^{\infty}$ is a subsequence of $\{x^k\}_{k=0}^{\infty}$ such that $\lim_{p\to\infty} x^{k_p} = x^*$, we have

$$\lim_{k\to\infty} \left\| x^k - x^* \right\| = \lim_{p\to\infty} \left\| x^{k_p} - x^* \right\| = 0$$

showing that any accumulation point x^* of $\{x^k\}_{k=0}^{\infty}$ is exactly the limit of $\{x^k\}_{k=0}^{\infty}$. ∎

The application of Theorem 4.1 depends on our ability to choose numbers M and r and a vector x^0 such that condition (4.8) is satisfied. We show below that this can be done when the functions f_i of the CFP (1.2) are quadratic or affine, and there is some a priori known ball which intersects Q. In actual applications it may be difficult to a priori decide whether the CFP (1.2) has or does not have solutions. However, as noted above, Algorithm 3.1 is well defined and will generate sequences $\{x^k\}_{k=0}^{\infty}$ no matter how the initial data M, r, and x^0 are chosen. This leads to the question whether it is possible to decide if Q is empty or not by simply analyzing the behavior of sequences $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 3.1. A partial answer to this question is contained in the following result.

COROLLARY 4.6. *Suppose that the CFP* (1.2) *has no solution, and that the envelope f is strictly convex. Then, no matter how the initial vector x^0 and the positive number M are chosen, any sequence* $\{x^k\}_{k=0}^{\infty}$ *generated by Algorithm* 3.1, *has the following properties:*

(i) *If* $\{x^k\}_{k=0}^{\infty}$ *is bounded and*

$$\lim_{k\to\infty} \left\| x^{k+1} - x^k \right\| = 0, \tag{4.35}$$

then f has a (necessarily unique) minimizer, and $\{x^k\}_{k=0}^{\infty}$ *converges to that minimizer, while*

$$\lim_{k\to\infty} f(x^k) = \inf\{f(x) \mid x \in R^n\}. \tag{4.36}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

(ii) *If f has no minimizer, then the sequence $\{x^k\}_{k=0}^{\infty}$ is unbounded, or the sequence $\{\|x^{k+1} - x^k\|\}_{k=0}^{\infty}$ does not converge to zero.*

Proof. Clearly, (ii) is a consequence of (i). In order to prove (i) observe that, since the CFP (1.2) has no solution, all values of f are positive. Also, if f has a minimizer, then this minimizer is unique because f is strictly convex.

If $\{x^k\}_{k=0}^{\infty}$ is bounded, then it has an accumulation point, say, x^*. By Lemma 4.3 there exists a sequence of positive integers $\{k_s\}_{s=0}^{\infty}$ such that (4.16) and (4.19)–(4.20) are satisfied. Using Lemma 4.3 again, we deduce that, if the limit λ_* in (4.16) is zero, then the vector $x^* = \lim_{s\to\infty} x^{k_s}$ is a solution of the CFP (1.2), i.e., $f(x^*) \leq 0$, contradicting the assumption that the CFP (1.2) has no solution. Hence, $\lambda_* > 0$. By (4.3), (4.35), and (4.16) we have that

$$0 = \lim_{s\to\infty} \lambda_{k_s} \nu^{k_s} = \lambda_* \nu^*. \tag{4.37}$$

Thus, we deduce that $\nu^* = 0$. From (4.19)–(4.20) and [15, Proposition 2.3.12] we obtain

$$0 = \nu^* = \sum_{i\in I(x^*)} w_i^* \xi_i^* \in \partial f(x^*) \tag{4.38}$$

showing that x^* is a minimizer of f. So, all of the accumulation points of $\{x^k\}_{k=0}^{\infty}$ coincide because f has no more than one minimizer. Consequently, the bounded sequence $\{x^k\}_{k=0}^{\infty}$ converges, and its limit is the unique minimizer of f. □

Remark 4.7. Checking numerically a condition such as (ii) in Corollary 4.6 or the condition in Corollary 4.9 below seems virtually impossible. But there is no escape from such situations in such mathematically oriented results. Condition (ii) in Corollary 4.6 is meaningful in the inconsistent case in which a feasible point does not exist, but a proximity function that "measures" the feasibility violation of the limit point can be minimized. An easy adaptation of the proof of Corollary 4.6 shows that, if the sequence $\{x^k\}_{k=0}^{\infty}$ has a bounded subsequence $\{x^{k_t}\}_{t=0}^{\infty}$ such that the limit $\lim_{t\to\infty}(x^{k_t+1} - x^{k_t}) = 0$, then all of the accumulation points of $\{x^{k_t}\}_{t=0}^{\infty}$ are the minimizers of f (even if f happens to be not strictly convex).

Remark 4.8. The fact that, for some choice of x^0 and M, a sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm 3.1, has the property that $\lim_{k\to\infty} f(x^k) = 0$ does not imply that the CFP (1.2) has a solution. For example, take in (1.2) $m = n = 1$ and $f_1(x) = e^{-x}$. Clearly, in this case (1.2) has no solution, and $f = f_1$. However, for $x^0 = 0$, $M = 1$, and $\lambda_k = (3/2)f(x^k)$, we have $\lim_{k\to\infty} f(x^k) = 0$.

A meaningful implication of Corollary 4.6 is the following result.

Corollary 4.9. *Suppose that the CFP* (1.2) *has no solution, and that f is strictly convex. Then, no matter how the initial vector x^0 and the positive number M are chosen in Algorithm* 3.1, *the following holds: If the series $\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|$ converges, then the function f has a unique global minimizer and the sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm* 3.1, *converges to that minimizer, while the sequence $\{f(x^k)\}_{k=0}^{\infty}$ converges to* $\inf\{f(x) \mid x \in R^n\}$.

Proof. When $\sum_{k=0}^{\infty} \|x^k - x^{k+1}\|$ converges to some number S we have

$$\left\|x^0 - x^{k+1}\right\| \leq \sum_{\ell=0}^{k} \left\|x^\ell - x^{\ell+1}\right\| \leq S \tag{4.39}$$

for all integers $k \geq 0$. This implies that the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded, and $\lim_{k\to\infty} \|x^k - x^{k+1}\| = 0$. Hence, by applying Corollary 4.6, we complete the proof. □

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Remark 4.10. Finding an initial vector x^0, the radius r and a positive number M satisfying condition $M \geq L(B(x^0, r))$ (and satisfying (4.2) provided that Q is nonempty) when there is no a priori knowledge about the existence of a solution of the CFP can be quite easily done when at least one of the sets Q_i, say Q_{i_0}, is bounded and the functions f_i are differentiable. In this case it is sufficient to determine a vector x^0 and a positive number r large enough so that the ball $B(x^0, r/2)$ contains Q_{i_0}. Clearly, for such a ball, if Q is nonempty, then condition (4.2) holds. Once the ball $B(x^0, r)$ is determined, finding a number $M \geq L(B(x^0, r))$ can be done by taking into account that the gradients of the differentiable convex functions $f_i : R^n \to R$ are necessarily continuous, and therefore the numbers

$$L_i = \sup\{\|\nabla f_i(x)\| \mid x \in B(x^0, r)\} \tag{4.40}$$

are necessarily finite. Since $L := \max\{L_i \mid 1 \leq i \leq m\}$ is necessarily a Lipschitz constant of f over $B(x^0, r)$, one can take $M = L$.

Remark 4.11. The method of choosing x^0, r, and M presented in Remark 4.10 does not require a priori knowledge of the existence of a solution of the CFP and can be applied even when Q is empty. In such a case one should compute, along the iterative procedure of Algorithm 3.1, the sums $S_k = \sum_{\ell=0}^{k} \|x^\ell - x^{\ell+1}\|$. Theorem 4.1 and Corollary 4.9 then provide the following insights and tools for solving the CFP, provided that f is strictly convex:

- If along the computational process the sequence S_k remains bounded from above by some number S^*, while the sequence $\{f(x^k)\}_{k=0}^{\infty}$ stabilizes itself asymptotically at some *positive* value, then the given CFP has no solution, but the sequence $\{x^k\}_{k=0}^{\infty}$ still approximates a global minimum of f, which may be taken as a surrogate solution of the given CFP.
- If along the computational process the sequence S_k remains bounded from above by some number S^*, while the sequence $\{f(x^k)\}_{k=0}^{\infty}$ stabilizes itself asymptotically at some *nonpositive* value, then the given CFP has a solution, and the sequence $\{x^k\}_{k=0}^{\infty}$ approximates such a solution.

5. Implementation of Algorithm 3.1 for linear or quadratic functions. The application of Algorithm 3.1 does not require a priori knowledge of the constant r. However, in order to implement this algorithm so that the conditions for convergence will be guaranteed, we have to determine numbers r and M as required by Theorem 4.1. The method proposed in Remark 4.10 might yield a very large value of r. This is due to the mathematical generality of Remark 4.10. The quadratic and affine cases treated next seem to be restrictive from the theoretical/mathematical point of view, but their importance lies in the fact that they cover many significant real-world applications.

We deal first with the problem of determining a number M such that

$$M \geq L(B(x^0, r)) \tag{5.1}$$

provided that an $r > 0$ is given. Recall that, if $g : R^n \to R$ is a continuously differentiable function, then by Taylor's formula, we have that, whenever $x, y \in B(x^0, r)$, there exists a $u \in [x, y]$ such that

$$\begin{aligned} \mid g(y) - g(x) \mid =\mid \langle \nabla g(u), y - x\rangle \mid &\leq \|\nabla g(u)\| \, \|y - x\| \\ &\leq \|y - x\| \max\{\|\nabla g(u)\| \mid u \in B(x^0, r)\}. \end{aligned} \tag{5.2}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

This shows that

$$\max\{\|\nabla g(u)\| \mid u \in B(x^0, r)\} \tag{5.3}$$

is a Lipschitz constant for g on $B(x^0, r)$. Suppose now that each function f_i is either linear or quadratic. Denote $I_1 = \{i \mid 1 \leq i \leq m,\ f_i \text{ is linear}\}$ and $I_2 = \{i \mid 1 \leq i \leq m,\ f_i \text{ is quadratic}\}$. Namely,

$$f_i(x) = \langle a^i, x\rangle + b_i \quad \text{for all} \quad i \in I_1, \tag{5.4}$$

with $a^i \in R^n \backslash \{0\}$ and $b_i \in R$, and

$$f_i(x) = \langle x, U_i x\rangle + \langle a^i, x\rangle + b_i \text{ for all } \ i \in I_2, \tag{5.5}$$

where $U_i = (u^i_{\ell,k})$ is a symmetric positive semidefinite $n \times n$ matrix, $a^i \in R^n$, and $b_i \in R$. We have, of course,

$$\nabla f_i(x) = \begin{cases} a^i & \text{if } \ i \in I_1, \\ 2U_i x + a^i & \text{if } \ i \in I_2, \end{cases} \tag{5.6}$$

so that (5.3) can give us Lipschitz constants for each f_i over $B(x^0, r)$. Denote

$$L_i := \begin{cases} \|a^i\| & \text{if } i \in I_1, \\ 2\,\|U_i\|_\infty \left(\|x^0\| + r\right) + \|a^i\| & \text{if } i \in I_2, \end{cases} \tag{5.7}$$

where $\|U_i\|_\infty$ is the operator norm of U_i. Due to (4.6), this implies that $\cup_{x\in B(x^0,r)}\partial f(x) \subseteq B(0, L)$, where

$$L := \max\{L_i \mid 1 \leq i \leq m\}. \tag{5.8}$$

Taking $\xi \in \partial f(x)$ and $\zeta \in \partial f(y)$, for some $x, y \in B(x^0, r)$, we have

$$\begin{aligned} L\,\|x-y\| &\geq \|\zeta\|\,\|x-y\| \geq \langle \zeta, y-x\rangle \geq f(y) - f(x) \\ &\geq \langle \xi, y-x\rangle \geq -\|\xi\|\,\|x-y\| \geq -L\,\|x-y\|\,, \end{aligned} \tag{5.9}$$

which implies that

$$|f(y) - f(x)| \leq L\,\|x-y\| \ \text{ for all } \ x, y \in B(x^0, r). \tag{5.10}$$

In other words, L is a Lipschitz constant of f over $B(x^0, r)$. Thus, given an $r > 0$, we can take M to be any number such that $M \geq L$. Note that choosing x^0 such that the corresponding r is small may speed up the computational process by reducing the number of iterations needed to reach a reasonably good approximate solution of the CFP. In general, determining a number r is straightforward when one has some information about the range of the variation of the coordinates of some solutions to the CFP.

For instance, if one knows a priori that the solutions of the CFP are vectors $x = (x_j)_{j=1}^n$ such that

$$\ell_j \leq x_j \leq u_j,\ 1 \leq j \leq n, \tag{5.11}$$

where, $\ell_j, u_j \in R$ for all j, then the set Q is contained in the hypercube of edge length $\delta = u_{\max} - \ell_{\min}$, whose faces are parallel to the axes of the coordinates, and centered at the point x^0 whose coordinates are $x^0_j = \frac{1}{2}(\ell_{\min} + u_{\max})$, where

$$\ell_{\min} := \min\{\ell_j \mid 1 \leq j \leq n\} \text{ and } u_{\max} := \max\{u_j \mid 1 \leq j \leq n\}. \tag{5.12}$$

Therefore, by choosing this x^0 as the initial point for Algorithm 3.1 and choosing $r = \sqrt{n}\delta$, condition (4.2) holds.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

6. Computational results. In this section, we compare the performance of Algorithms 2.2, 2.4, and 3.1 by examining a few test problems. There are a number of degrees-of-freedom used to evaluate and compare the performance of the algorithms. These are the maximum number of iterations, the number of constraints, the lower and upper bounds of the box constraints, the values of the relaxation parameters, the initial values of the steering parameters, and the steering sequence. In all our experiments, the steering sequence of Algorithm 2.4 assumed the form

$$\sigma_k = \frac{\sigma}{k+1}, \tag{6.1}$$

with a fixed user-chosen constant σ. The main performance measure is the value of $f(x^k)$ plotted as a function of the iteration index k.

6.1. Test problem description. There are three types of constraints in our test problems: box constraints, linear constraints, and quadratic constraints. Some of the numerical values used to generate the constraints are uniformly distributed random numbers lying in the interval $\tau = [\tau_1, \tau_2]$, where τ_1 and τ_2 are user-chosen predetermined values.

The n box constraints are defined by

$$\ell_j \leq x_j \leq u_j, \quad j = 1, 2, \ldots, n, \tag{6.2}$$

where $\ell_j, u_j \in \tau$ are the lower and upper bounds, respectively. Each of the N_q quadratic constrains is generated according to

$$G_i(x) = \langle x, U_i x\rangle + \langle v^i, x\rangle + \beta_i, \quad i = 1, 2, \ldots, N_q. \tag{6.3}$$

Here U_i are the $n \times n$ matrices defined by

$$U_i = W_i \Lambda_i W_i^T, \tag{6.4}$$

where the $n \times n$ matrices Λ_i are diagonal and positive definite, given by

$$\Lambda_i = \operatorname{diag}\left(\delta_1^i,\, \delta_2^i, \ldots, \delta_n^i\right), \tag{6.5}$$

where $0 < \delta_1^i \leq \delta_2^i \leq \cdots \leq \delta_n^i \in \tau$ are generated randomly. The matrices W_i are generated by orthonormalizing an $n \times n$ random matrix whose entries lie in the interval τ. Finally, the vector $v^i \in R^n$ is constructed so that all of its components lie in the interval τ, and similarly, the scalar $\beta_i \in \tau$. The N_ℓ linear constraints are constructed in a similar manner according to

$$L_i(x) = \langle y^i, x\rangle + \gamma_i, \quad i = 1, 2, \ldots, N_\ell. \tag{6.6}$$

Thus, the total number of constraints is $n + N_q + N_\ell$.

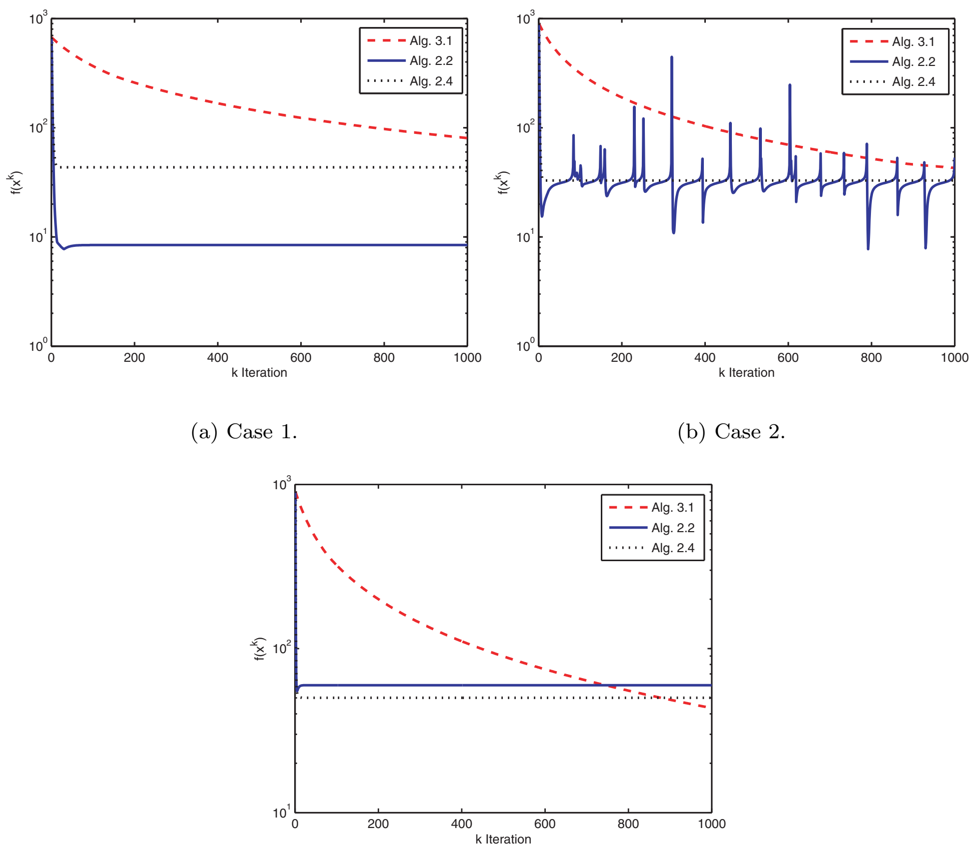
Table 6.1 summarizes the test cases used to evaluate and compare the performance of Algorithms 2.2, 2.4, and 3.1. In these eight experiments, we modified the value of the constant σ in (6.1), the interval τ, the number of constraints, the number of iterations, and the relative tolerance ε, used as a termination criterion between subsequent iterations.

In Table 6.1, Cases 1 and 2 represent small-scale problems with a total of 13 constraints, whereas Cases 4–6 represent midscale problems with a total of 130 constraints. Cases 6–8 examine the case of overrelaxation, wherein the initial steering (relaxation) parameter is at least 2.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Table 6.1
Test cases for performance evaluation.

Case	$\alpha/\sigma/\lambda$	τ	n	N_q	N_ℓ	Iterations	ε
1	1.1		3	5	5	1,000	
2	1.1		3	5	5	1,000	
3	1.98		3	5	5	1,000	
4	1.98	$[-0.1, 0.1]$	30	50	50	1,000	0.1
5	1.98	$[-10, 10]$	30	50	50	100,000	0.1
6	2	$[-0.1, 0.1]$	30	50	50	1,000	0.1
7	3	$[-10, 10]$	3	5	5	1,000	0.1
8	5	$[-0.1, 0.1]$	3	5	5	1,000	0.1



(a) Case 1.

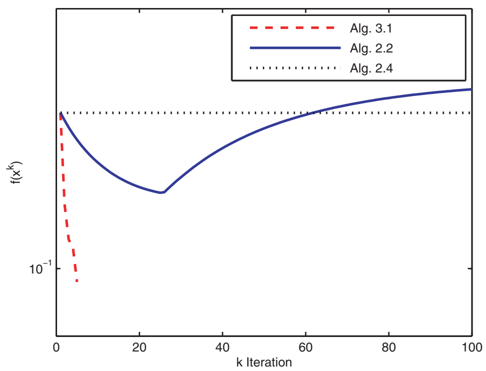
(b) Case 2.

(c) Case 3.

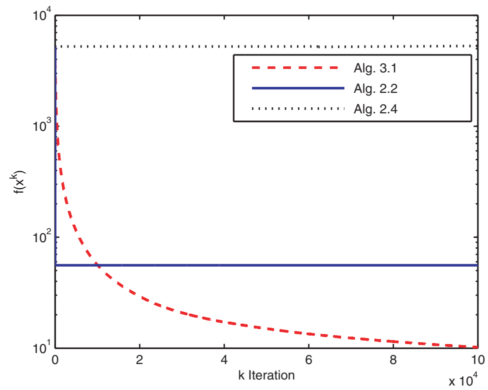
Fig. 1. *Simulation results for a small-scale problem, comparing Algorithms* 2.2, 2.4, *and* 3.1.

6.2. Results. The results of our experiments are depicted in Figures 1–3. The results of Cases 1–3 are shown in Figures 1(a)–1(c), respectively. It is seen that, in Case 1, Algorithm 2.2 has better initial convergence than Algorithms 2.4 and 3.1. However, in Case 2, Algorithm 2.4 yields fast and smooth initial behavior, while Algorithm 2.2 oscillates chaotically. Algorithm 3.1 exhibits slow initial convergence, similarly to Case 1. In Case 3, Algorithm 3.1 supersedes the performance of the other two algorithms, since it continues to converge toward zero. However, none of the algo-

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

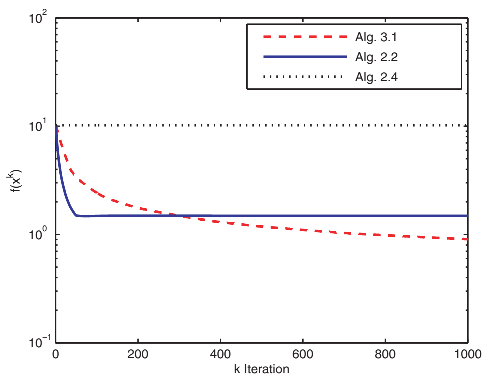


(a) Case 4.

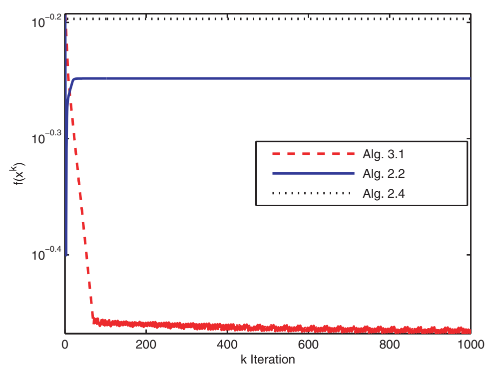


(b) Case 5.

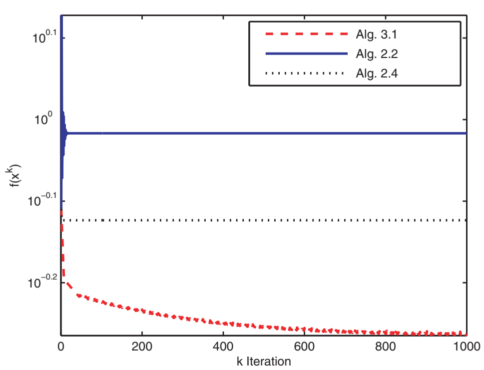
FIG. 2. *Simulation results for a midscale problem, comparing Algorithms* 2.2*,* 2.4*, and* 3.1*.*



(a) Case 6.



(b) Case 7.



(c) Case 8.

FIG. 3. *Simulation results for small-scale and midscale problems with overrelaxation, comparing Algorithms* 2.2*,* 2.4*, and* 3.1*.*

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

rithms detects a feasible solution, since none converged to the tolerance threshold after the maximum number of iterations.

The midsized problems of Cases 4 and 5 are depicted by Figures 2(a) and 2(b). Figure 2(a) shows that Algorithm 3.1 detects a feasible solution, while both Algorithms 2.2 and 2.4 fail to detect such a solution. The curve of Algorithm 3.1 in Figure 2(a) stops when it reaches the feasible point detection tolerance, which is 0.1. Once the point is detected, there is no need to further iterate, and the process stops. The curve for Algorithm 2.2 in this figure shows irregular behavior since it searches for a feasible solution without reaching the detection threshold of 0.1, and accumulated numerical errors start to affect it. Figure 2(b) shows a phenomenon similar to the one observed in the small-scale problem: Algorithm 3.1 continues to seek a feasible solution, while Algorithms 2.2 and 2.4 converge to a steady state, indicating the failure to detect a feasible solution.

In the experiments Cases 6–8, Algorithm 3.1 outperforms the other algorithms, arriving very close to finding feasible solutions. It should be observed that the behavior of Algorithm 3.1 observed above is the result of the way in which the relaxation parameters λ_k are self-regulating their sizes. In Algorithm 3.1 the relaxation parameter λ_k can be chosen (see (3.6)) to be any number of the form

$$\lambda_k = \beta_k \frac{\max(0, f(x^k))}{M^2} + 2(1-\beta_k)\frac{\max(0, f(x^k))}{M^2} = (2-\beta_k)\frac{\max(0, f(x^k))}{M^2}, \tag{6.7}$$

where β_k runs over the interval $[0, 1]$. Consequently, the size of λ_k can be very close to zero when x^k is close to a feasible solution (no matter how β_k is chosen in $[0, 1]$). Also, λ_k may happen to be much larger than 2 when x^k is far from a feasible solution, and the number $f(x^k)$ is large enough (note that $2 - \beta_k$ stays between 1 and 2). So, Algorithm 3.1 is naturally underrelaxing or overrelaxing the computational process according to the relative position of the current iterate x^k to the feasibility set of the problem. As our experiments show, in some circumstances, this makes Algorithm 3.1 behave better than the other procedures we compare with it. At the same time, the self-regulation of the relaxation parameters, which is essential in Algorithm 3.1, may happen to reduce the initial speed of convergence of this procedure, that is, Algorithm 3.1 may require more computational steps in order to reach a point x^k, which is close enough to the feasibility set such that its self-regulatory features are to be really advantageous for providing a very precise solution of the given problem (which the other procedures may fail to do since they may become stationary in the vicinity of the feasibility set). Another interesting feature of Algorithm 3.1, which differentiates it from the other algorithms we compare it with, is its essentially nonsimultaneous character: Algorithm 3.1 does not necessarily ask for $w_i^k > 0$ for all $i \in \{1, \dots, m\}$. The set of positive weights w_i^k, which condition the progress of the algorithm at step k, essentially depends on the current iterate x^k (see (3.4)) and allows reducing the number of subgradients needed to be computed at each iterative step (in fact, one can content himself with only one $w_i^k > 0$, and thus with a single subgradient ξ_i^k). This may be advantageous in cases when computing subgradients is difficult and therefore time consuming.

The main observations can be summarized as follows:

1. Algorithm 3.1 exhibits faster initial convergence than the other algorithms in the vicinity of points with very small $f(x^k)$. When the algorithms reach points with small $f(x^k)$ values, then Algorithm 3.1 tends to further reduce the value of $f(x^k)$, while the other algorithms tend to converge onto a constant steady-state value.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

2. The problem dimensions in our experiments have little impact on the behavior of the algorithms.
3. All the examined small-scale problems have no feasible solutions. This can be seen from the fact that all three algorithms stabilize around $f(x^k) = 50$.
4. The chaotic oscillations of Algorithm 2.2 in the underrelaxed case is due to the fact that this algorithm has no internal mechanism to self-adapt its progress to the distance between the current iterates and the sets whose intersections are to be found. This phenomenon can hardly happen in Algorithm 3.1 because its relaxation parameters are self-adapting to the size of the current difference between successive iterations. This is an important feature of this algorithm. However, this feature also renders it somewhat slower than the other algorithms.
5. In some cases, Algorithms 2.2 and 2.4 indicate that the problem has no solution. In contrast, Algorithm 3.1 continues to make progress and seems to indicate that the problem has a feasible solution. This phenomenon is again due to the self-adaptation mechanism and can be interpreted in one of the following ways: (a) The problem indeed has a solution, but Algorithms 2.2 and 2.4 are unable to detect it (because they stabilize too fast). Algorithm 3.1 detects a solution provided that it is given enough running time; (b) The problem has no solution, and then Algorithm 3.1 will stabilize close to zero, indicating that the problem has no solution, but this may be due to computing (round-off) errors. Thus, a very small perturbation of the functions involved in the problem may render the problem feasible.

7. Conclusions. We have studied here mathematically and experimentally subgradient projections methods for the convex feasibility problem. The behavior of the fully simultaneous subgradient projections method in the inconsistent case is not known. Therefore, we studied and tested two options. One is the use of steering parameters instead of relaxation parameters, and the other is a variable relaxation strategy, which is self-adapting. Our small-scale and midscale experiments are not, decisive in all aspects and call for further research. But one general feature of the algorithm with the self-adapting strategical relaxation is its stability (nonoscillatory) behavior, and its relentless improvement of the iterations towards a solution in all cases. At this time we have not yet refined enough our experimental setup. For example, by the iteration index k on the horizontal axes of our plots we consider a whole sweep through all the sets of the convex feasibility problem, regardless of the algorithm. This is a good first approximation by which to compare the different algorithms. More accurate comparisons should use actual run times. Also, several numerical questions still remain unanswered in this report. These include the effect of various values of the constant σ as well as algorithmic behavior for higher iteration indices. In light of the applications mentioned in section 1, higher dimensional problems must be included. These and other computational questions are currently investigated.

Acknowledgments. We gratefully acknowledge the constructive comments of two anonymous referees, which helped us improve an earlier version of this paper.

REFERENCES

[1] H.H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl., 202 (1996), pp. 150–159.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

[2] H.H. Bauschke and J.M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Rev., 38 (1996), pp. 367–426.
[3] H.H. Bauschke, J.M. Borwein, and A.S. Lewis, *The method of cyclic projections for closed convex sets in Hilbert space*, in Recent Developments in Optimization Theory and Nonlinear Analysis, Contemp. Math., 204, Y. Censor and S. Reich, eds., 1997, pp. 1–38.
[4] H.H. Bauschke, P.L. Combettes, and S.G. Kruk, *Extrapolation algorithm for affine-convex feasibility problems*, Numer. Algorithms, 41 (2006), pp. 239–274.
[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.
[6] D. Butnariu and A.N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
[7] D. Butnariu and A. Mehrez, *Convergence criteria for generalized gradient methods of solving locally Lipschitz feasibility problems*, Comput. Optim. Appl., 1 (1992), pp. 307–326.
[8] D. Butnariu and E. Resmerita, *Averaged subgradient methods for optimization and Nash equilibria computation*, Optimization, 51 (2002), pp. 863–888.
[9] Y. Censor, *Mathematical optimization for the inverse problem of intensity modulated radiation therapy*, in Intensity-Modulated Radiation Therapy: The State of the Art, American Association of Physicists in Medicine, Medical Physics Monograph 29, J.R. Palta and T.R. Mackie, eds., Medical Physics Publishing, Madison, Wisconsin, 2003, pp. 25–49.
[10] Y. Censor, *Computational acceleration of projection algorithms for the linear best approximation problem*, Linear Algebra Appl., 416 (2006), pp. 111–123.
[11] Y. Censor, M.D. Altschuler, and W.D. Powlis, *On the use of Cimmino's simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning*, Inverse Problems, 4 (1988), pp. 607–623.
[12] Y. Censor and A. Lent, *Cyclic subgradient projections*, Math. Program., 24 (1982), pp. 233–235.
[13] Y. Censor and S.A. Zenios, *Parallel Optimization: Theory, Algorithms, and Applications*, Oxford University Press, New York, 1997.
[14] J.W. Chinneck, *The constraint consensus method for finding approximately feasible points in nonlinear programs*, INFORMS J. Comput., 16 (2004), pp. 255–265.
[15] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.
[16] P.L. Combettes, *The foundations of set-theoretic estimation*, Proc. IEEE, 81 (1993), pp. 182–208.
[17] P.L. Combettes, *Inconsistent signal feasibility problems: Least-squares solutions in a product space*, IEEE Trans. Signal Process., SP-42 (1994), pp. 2955–2966.
[18] P.L. Combettes, *The convex feasibility problem in image recovery*, Adv. Imaging and Electron Phys., 95 (1996), pp. 155–270.
[19] P.L. Combettes, *Convex set theoretic image recovery by extrapolated iterations of parallel subgradient projections*, IEEE Trans. Image Process., 6 (1997), pp. 493–506.
[20] G. Crombez, *Non-monotoneous parallel iteration for solving convex feasibility problems*, Kybernetika, 39 (2003), pp. 547–560.
[21] G. Crombez, *A sequential iteration algorithm with non-monotoneous behavior in the method of projections onto convex sets*, Czechoslovak Math. J., 56 (2006), pp. 491–506.
[22] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York, 2001.
[23] L.T. Dos Santos, *A parallel subgradient method for the convex feasibility problem*, J. Comput. Appl. Math., 18 (1987), pp. 307–320.
[24] I.I. Eremin, *Fejér mappings and convex programming*, Siberian Math. J., 10 (1969), pp. 762–772.
[25] L. Gubin, B. Polyak, and E. Raik, *The method of projections for finding the common point of convex sets*, USSR Comput. Math. Math. Phys., 7 (1967), pp. 1–24.
[26] G.T. Herman, *Image Reconstruction From Projections: The Fundamentals of Computerized Tomography*, Academic Press, New York, 1980.
[27] G.T. Herman and L.B. Meyer, *Algebraic reconstruction techniques can be made computationally efficient*, IEEE Trans. Medical Image, 12 (1993), pp. 600–609.
[28] A.N. Iusem and A.R. De Pierro, *Convergence results for an accelerated nonlinear Cimmino algorithm*, Numer. Math., 49 (1986), pp. 367–378.
[29] K.C. Kiwiel, *The efficiency of subgradient projection methods for convex optimization* I: *General level methods*, SIAM J. Control Optim., 34 (1996), pp. 660–676.
[30] K.C. Kiwiel, *The efficiency of subgradient projection methods for convex optimization* II: *Implementations and extensions*, SIAM J. Control Optim., 34 (1996), pp. 677–697.
[31] K.C. Kiwiel and B. Lopuch, *Surrogate projection methods for finding fixed points of firmly nonexpansive mappings*, SIAM J. Optim., 7 (1997), pp. 1084–1102.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

[32] A.N. Iusem and L. Moledo, *A finitely convergent method of simultaneous subgradient projections for the convex feasibility problem*, Comput. Appl. Math., 5 (1986), pp. 169–184.
[33] L.D. Marks, W. Sinkler, and E. Landree, *A feasible set approach to the crystallographic phase problem*, Acta Crystallogr., A55 (1999), pp. 601–612.
[34] M. A. Noor, *Some developments in general variational inequalities*, Appl. Math. Comput., 152 (2004), pp. 197–277.
[35] D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Sijthoff & Noordhoff, Alphen aan den Rijn, the Netherlands, 1978.
[36] G. Pierra, *Decomposition through formalization in a product space*, Math. Program., 28 (1984), pp. 96–115.
[37] R.R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Springer-Verlag, Berlin, 1993.
[38] I. Yamada, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim., 25 (2004), pp. 619–655.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.