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Semi-Analytical Method for Calculating the Elliptic Restricted Three-Body Problem Monodromy Matrix

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Introduction

RECENT years have seen a rising interest in launching unmanned spacecraft into libration-point orbits for scientific missions. Past missions such as ISEE-3, SOHO, and Genesis successfully used quasiperiodic orbits about the L_1 and L_2 collinear libration points of the sun–Earth system. With the embarkation of future NASA and ESA missions such as Darwin, Terrestrial Planet Finder (TPF), and Single Aperture Far Infrared Observatory (SAFIR), to be launched into libration-point orbits, there is an opportunity to design and evaluate novel trajectory planning, simulation, and control schemes.

In practical analysis and design of missions around libration points, the circular restricted three-body problem (CR3BP) model is usually adopted [1–4]. Although this model has proven fruitful, it possesses an inherent approximation, assuming that the orbits of the primaries are circular. However, both the motion of the Earth around the sun and the motion of moon around the Earth are eccentric. Incorporating the eccentricity term into the equations of motion renders a more general model, known as the elliptic restricted three-body problem (ER3BP). The ER3BP has significant topological differences compared with the CR3BP. For example, the position of the libration points is not constant, but rather pulsating with respect to Earth. Moreover, the Jacobi integral is time- (true-anomaly-) dependent.

Several works have addressed the problem of finding natural periodic orbits in the planar ER3BP based on specialized regularizations [5,6]. Derivation of such orbits through numerical searches was accomplished in [7,8]. Almost all methods for calculating orbits about the collinear libration points require the monodromy matrix, which is the state-transition matrix evaluated at the orbital period. Moreover, design of stationkeeping maneuvers, whether impulsive [9] or continuous [10], also requires the monodromy matrix.

Although the evaluation of the monodromy matrix in the CR3BP is trivial, the problem is more involved in the more general setup of the ER3BP, because the linearized system is nonautonomous. In fact, the linearized system is linear parameter-varying (LPV) and periodic, exhibiting implicit dependence upon time through the true anomaly. It is therefore important to develop reliable algorithms for computing the monodromy matrix.

In this work, we develop a new, semianalytical method for calculation of the monodromy matrix. We propose to expand the state-transition matrix into orthogonal Chebyshev polynomials of the first and second kind, shifted to fit the time interval in use. The Chebyshev approximation transforms the nonautonomous differential equations required to calculate the state-transition matrix into a set of algebraic equations. This approach can be used as a computationally efficient scheme for computing the state-transition matrix and as a simple check for verifying the accuracy of monodromy matrix calculations using a direct numerical integration.

Equations of Motion

The ER3BP is a dynamical model that describes the motion of an infinitesimal-mass body, a space vehicle, under the gravitational influence of two massive gravitational bodies, the primaries. The most popular coordinate system used to model the ER3BP dynamics has its origin set at the barycenter of the large primary M_1 and the small primary M_2. The x axis is positive in the direction of M_2, the z axis is perpendicular to the plane of rotation and is positive upwards, and the y axis completes the setup to yield a Cartesian, rectangular, dextral rotating reference frame. The small primary is orbiting the large primary on an elliptic orbit with eccentricity e. This orbit complies with the two-body Keplerian motion; the distance between the primaries ρ depends upon the true anomaly f through the conic equation

$$\rho = \frac{p}{1 + e\cos f} \tag{1}$$

where p is the semilatus rectum. The position vector of the spacecraft in the rotating barycentric frame is $\boldsymbol{R} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$. By defining $\mu \triangleq M_2/(M_1 + M_2)$, we can write the equations of motion as [6]

$$\begin{aligned}
\ddot{X} - \dot{f}^2 X - 2\dot{f}\dot{Y} - \ddot{f}Y &= -\frac{(1-\mu)(X+\rho)}{[(X+\mu\rho)^2 + Y^2 + Z^2]^{3/2}} \\
&\quad - \frac{\mu[X + (\mu-1)\rho]}{\{[X+(\mu-1)\rho]^2 + Y^2 + Z^2\}^{3/2}} \\
\ddot{Y} - \dot{f}^2 Y - 2\dot{f}\dot{X} - \ddot{f}X &= -\frac{(1-\mu)Y}{[(X+\mu\rho)^2 + Y^2 + Z^2]^{3/2}} \\
&\quad - \frac{\mu Y}{\{[X+(\mu-1)\rho]^2 + Y^2 + Z^2\}^{3/2}} \\
\ddot{Z} &= -\frac{(1-\mu)Z}{[(X+\mu\rho)^2 + Y^2 + Z^2]^{3/2}} \\
&\quad - \frac{\mu Z}{\{[X+(\mu-1)\rho]^2 + Y^2 + Z^2\}^{3/2}}
\end{aligned} \tag{2}$$

To simplify Eqs. (2), a transformation to rotating–pulsating coordinates is required [11]. Defining $X = \rho\xi$, $Y = \rho\eta$, $Z = \rho\zeta$, and transforming time derivatives into derivatives with respect to true anomaly, yield the compact equations

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$$\xi'' - 2\eta' = \frac{\partial\Omega}{\partial\xi}, \qquad \eta'' + 2\xi' = \frac{\partial\Omega}{\partial\eta}, \qquad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{3}$$

where Ω is the pseudopotential function

$$\Omega = \frac{1}{1+e\cos f}\left[\frac{1}{2}(\xi^2+\eta^2-e\zeta^2\cos f) + \frac{(1-\mu)}{\sqrt{(\xi+\mu)^2+\eta^2+\zeta^2}} + \frac{\mu}{\sqrt{(\xi+\mu-1)^2+\eta^2+\zeta^2}}\right] \tag{4}$$

Emergence of the Monodromy Matrix in the Linearized Model

The location of the libration points of system in rotating–pulsating coordinates is constant and identical to their location in the CR3BP setup. It is therefore straightforward to perform linearization of the equations of motion about the libration points. To that end, we define a state vector as

$$\begin{aligned} \boldsymbol{x} &= [\delta\xi \quad \delta\eta \quad \delta\zeta \quad \delta\xi' \quad \delta\eta' \quad \delta\zeta']^T \\ &= [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6]^T \end{aligned} \tag{5}$$

For an initial true anomaly f_I, the linearized equations of motion assume a periodic LPV state-space representation of the form

$$\boldsymbol{x}'(f) = \boldsymbol{A}(f)\boldsymbol{x}(f), \qquad \boldsymbol{x}(f_I) = \boldsymbol{x}_I, \qquad \boldsymbol{A}(f) = \boldsymbol{A}(f+T) \tag{6}$$

with a period $T = 2\pi$. The matrix $\boldsymbol{A}$ satisfies

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \\ \boldsymbol{A_{21}} & \boldsymbol{A_{22}} \end{bmatrix}, \\ \boldsymbol{A}_{21} &= \frac{1}{1+e\cos(f)}\begin{bmatrix} a_{41} & 0 & 0 \\ 0 & a_{52} & 0 \\ 0 & 0 & a_{63}-e\cos(f) \end{bmatrix}, \\ \boldsymbol{A_{22}} &= \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{7}$$

where the values of the constants a_{41}, a_{52}, a_{63} depend upon the location of the three collinear points. It is well known [11] that the linear approximation about the collinear points in the CR3BP is unstable. However, the linear CR3BP is autonomous, whereas the linear ER3BP is parameter (time) varying. Thus, to study the stability of the linearized ER3BP, we must adopt Floquet's theory [12], dedicated to periodic LPV system.

The main result of Floquet's theory shows that the stability of periodic LPV systems can be deduced from analyzing the stability of an equivalent, time-invariant, system. Letting $\boldsymbol{\Psi}(f, f_I)$ denote the state-transition matrix, Floquet's equivalent system is obtained from the transformation

$$z(f) = \boldsymbol{P}(f)\boldsymbol{x}(f), \qquad \boldsymbol{P}(f) = e^{\boldsymbol{J}f}\boldsymbol{\Psi}^{-1}(f, f_I) \tag{8}$$

The equivalent time-invariant system is then simply

$$z'(f) = \boldsymbol{J}z(f) \tag{9}$$

where the Floquet matrix $\boldsymbol{J}$ is defined as

$$\boldsymbol{J} = \frac{1}{T}\ln\boldsymbol{\Phi} \tag{10}$$

and $\boldsymbol{\Phi}$ is the monodromy matrix, given by

$$\boldsymbol{\Phi} = \boldsymbol{\Psi}(f = f_I + T, f_I) \tag{11}$$

The monodromy matrix can be calculated in several ways, the most common being a direct numerical integration of the state-transition matrix through the matrix differential equations

$$\frac{\mathrm{d}}{\mathrm{d}f}\boldsymbol{\Psi}(f, f_I) = \boldsymbol{A}(f)\boldsymbol{\Psi}(f, f_I) \qquad \boldsymbol{\Psi}(f, f_I) = \boldsymbol{I} \tag{12}$$

Although a direct numerical integration is straightforward, it can involve considerable numerical errors over long time scales. It is therefore desired to develop a semianalytical approximation for the monodromy matrix entries verifying the numerical integration.

In this work, we calculate the monodromy matrix by developing a semianalytical approximation based on orthogonal Chebyshev polynomials [13–15]. The following section elaborates on the semianalytical procedure for calculation of the monodromy matrix entries.

Monodromy Matrix Approximation via Chebyshev Polynomials

Chebyshev polynomials are orthogonal polynomials characterized by a few useful properties. Most notably, derivatives, integrals, and products of Chebyshev polynomials can be conveniently expressed via simple manipulations of their coefficients.

Chebyshev polynomials of the first kind $V(\phi)$ and the second kind $U(\phi)$ are defined, respectively, by the identities

$$\begin{aligned} &V_r(\phi) = \cos(rf), \qquad U_r(\phi) = \sin[(r+1)f]/\sin f, \\ &0 \le f \le \pi, \qquad r = 0, 1, 2, \ldots \end{aligned} \tag{13}$$

where $\phi = \cos(f) \in [-1, 1]$. To approximate the entries of the monodromy matrix, we must perform the dilation–translation transformation

$$\phi^* = (\phi + 1)/2 \tag{14}$$

so that in the modified coordinates, $\phi^* \in [0, 1]$. The shifted Chebyshev polynomials of the first kind, written in recursive form, are given by

$$\begin{aligned} &V_0^* = 1, \qquad V_1^* = 2\phi^* - 1, \\ &V_{r+1}^* = 2(2\phi^* - 1)V_r^* - V_{r-1}^*, \qquad \phi^* \in [0, 1] \end{aligned} \tag{15}$$

whereas the shifted Chebyshev polynomials of the second kind are written as

$$\begin{aligned} &U_0^* = 1, \qquad U_1^* = 2(2\phi^* - 1), \\ &U_{r+1}^* = 2(2\phi^* - 1)U_r^* - U_{r-1}^*, \qquad \phi^*[0, 1] \end{aligned} \tag{16}$$

Chebyshev polynomials of both kinds are orthogonal; the shifted polynomials of the first kind are orthogonal with respect to the weight function

$$w_V(\phi^*) = (\phi^* - \phi^{*2})^{-1/2} \tag{17}$$

so that

$$\int_0^1 V_r^*(\phi^*)V_k^*(\phi^*)w_V(\phi^*)\,\mathrm{d}\phi^* = \begin{cases} 0, & r \ne k \\ \pi/2, & r = k \ne 0 \\ \pi, & r = k = 0 \end{cases} \tag{18}$$

whereas the polynomials of the second kind are orthogonal with respect to the weight function

$$w_U(\phi^*) = (\phi^* - \phi^{*2})^{1/2} \tag{19}$$

satisfying

$$\int_0^1 U_r^*(\phi^*)U_k^*(\phi^*)w_U(\phi^*)\,\mathrm{d}\phi^* = \begin{cases} 0, & r \ne k \\ \pi/8, & r = k \end{cases} \tag{20}$$

To derive semianalytical expressions for the monodromy matrix, we write $\phi^* = (1 + \cos f)/2$ [cf. (14)] and expand each true-anomaly-dependent entry $a_{ij}(\phi^*)$ of the submatrix $\boldsymbol{A}_{21}$ [cf. (7)] into Chebyshev polynomials. We repeat the same procedure for each component x_i, $i = 1, \ldots, 6$ of the state $\boldsymbol{x}$ of system (6), and then use the special properties of Chebyshev polynomials to obtain a semianalytical approximation to the transition and monodromy matrices.

The m-1-order expansion of any function $g(\tau)$, $\tau \in [0,1]$ into Chebyshev polynomials can be written as

$$g(\tau) = \sum_{r=0}^{m-1} \kappa_r^i S_r^*(\tau) \tag{21}$$

where κ_r is determined by the quadrature

$$\kappa_r = \frac{1}{\delta}\int_0^1 w(\tau)g(\tau)S_r^*(\tau)\,\mathrm{d}\tau, \qquad r = 0,1,2,\ldots \tag{22}$$

In Eqs. (21) and (22), $w = w_V$ and $S_r^* = V_r^*$ for polynomials of the first kind, and $w = w_U$ and $S_r^* = U_r^*$ for polynomials of the second kind. The constant δ satisfies

$$\begin{aligned} \delta &= \begin{cases} \pi/2, & r \neq 0 \\ \pi, & r = 0 \end{cases} \qquad \text{for } S_r^* = V_r^* \\ \delta &= \pi/8, \qquad r = 0,1,2,\ldots \quad \text{for } S_r^* = U_r^* \end{aligned} \tag{23}$$

By applying Eqs. (21) and (22), on each component of the state vector x_i, we get

$$\begin{aligned} x_i &= \sum_{r=0}^{m-1} b_r^i S_r^* = \{\boldsymbol{S}^*\}^T \boldsymbol{b}^i, \qquad i = 1,\ldots,6, \\ \boldsymbol{b}^i &= [b_0^i, b_1^i, \cdots, b_{m-1}^i]^T, \qquad \{\boldsymbol{S}^*\} = \{S_0^{*i}, S_1^{*i}, \ldots, S_{m-1}^{*i}\}^T \end{aligned} \tag{24}$$

where $\boldsymbol{b}^i$ is a yet unknown Chebyshev polynomial coefficient vector. In a similar manner, we can calculate a Chebyshev polynomial coefficient vector for each (true-anomaly-dependent) entry a_{ij} of $\boldsymbol{A}_{21}$, wherefrom

$$\begin{aligned} a_{ij} &= \sum_{r=0}^{m-1} d_r^{ij} S_r^* = \{\boldsymbol{S}^*\}^T \boldsymbol{d}^{ij}, \qquad i,j = 1,\ldots,6, \\ \boldsymbol{d}^{ij} &= [d_0^{ij}, d_1^{ij}, \cdots, d_{\mathrm{m}-1}^{ij}]^T \end{aligned} \tag{25}$$

where $\boldsymbol{d}^{ij}$ are polynomial coefficients calculated according to the quadrature (22).

We can now use the unique properties of Chebyshev polynomials, that is, express the product of any two polynomials using the *product operational matrix* [13], and the quadrature of shifted Chebyshev polynomials using the *integration operational matrix* [13]. This formalism transforms the problem of solving the vector differential Eq. (6) into the following system of algebraic equations:

$$[\hat{S}(t)]^T \boldsymbol{b} - [\hat{S}(t)]^T \boldsymbol{x}_I = [\hat{S}(t)]^T \boldsymbol{PB} + [\hat{S}(t)]^T \boldsymbol{RB} \tag{26}$$

In (26), $\boldsymbol{b} = [(\boldsymbol{b}^1)^T, \ldots, (\boldsymbol{b}^6)^T]$ is a $6m$-dimensional column vector of unknown polynomial coefficients [cf. (24)], and

$$[\hat{S}(t)]^T = \boldsymbol{I}_6 \otimes \{\boldsymbol{S}^*(t)\}^T \tag{27}$$

where $\otimes$ denotes the Kronecker product. The matrix $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{0}_{3m\times 3m} & \hat{\boldsymbol{G}} \\ \boldsymbol{0}_{3m\times 3m} & \hat{\boldsymbol{C}} \end{bmatrix} \tag{28}$$

where

$$\hat{\boldsymbol{G}} = \boldsymbol{I}_3 \otimes \bar{\boldsymbol{G}}^T, \qquad \hat{\boldsymbol{C}} = \boldsymbol{A}_{22} \otimes \bar{\boldsymbol{G}}^T \tag{29}$$

and $\bar{\boldsymbol{G}}^T$ is the $3m \times 3m$ integration operational matrix, whose explicit form can be found in [13]. The matrix $\boldsymbol{R}$ assumes the form

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{0}_{3m\times 3m} & \boldsymbol{0}_{3m\times 3m} \\ \hat{\boldsymbol{R}}^* & \hat{\boldsymbol{C}}^* \end{bmatrix} \tag{30}$$

with the submatrices

$$\hat{\boldsymbol{R}}^* = \hat{\boldsymbol{G}}\boldsymbol{Q}_{21}, \qquad \hat{\boldsymbol{C}}^* = \hat{\boldsymbol{G}}\boldsymbol{Q}_{22} \tag{31}$$

where $\boldsymbol{Q}_{21}$ and $\boldsymbol{Q}_{22}$ are the $3m \times 3m$ product operational matrices [13], whose entries depend upon $\boldsymbol{d}^{ij}$ [cf. Equation (25)]. Finally, we choose

$$\boldsymbol{x}_I = \boldsymbol{I}_6 \otimes [1, \boldsymbol{0}_{1\times m-1}]^T \tag{32}$$

and calculate the monodromy matrix through

Table 1 Numerical values for the coefficients of the linearized system

	a_{41}	a_{52}	a_{63}
L_1	3.021383283	−0.01069164180	−1.010691642
L_2	11.29521865	−4.147609411	−5.147609411
L_3	7.380834356	−2.190417208	−3.190417208

Table 2 The error matrix norm comparing the calculation of the monodromy matrix semianalytically and numerically

	$m = 14$		$m = 16$	
$\|\boldsymbol{E}\|$	First kind	Second kind	First kind	Second kind
L_1	$1.8957 \cdot 10^{-7}$	$1.4411 \cdot 10^{-6}$	$2.41 \cdot 10^{-8}$	$1.7259 \cdot 10^{-7}$
L_2	$1.1504 \cdot 10^{-6}$	$2.0128 \cdot 10^{-5}$	$1.2351 \cdot 10^{-7}$	$2.4109 \cdot 10^{-6}$
L_3	$7.5185 \cdot 10^{7}$	$7.3383 \cdot 10^{-6}$	$6.2247 \cdot 10^{-8}$	$9.1814 \cdot 10^{-7}$

Table 3 The error matrix norm comparing between the semianalytical calculation of the state transition and the closed-form analytic solution for the CR3BP case

	$m = 14$		$m = 16$	
$\|\boldsymbol{E}\|$	First kind	Second kind	First kind	Second kind
L_1	$1.1037 \cdot 10^{-8}$	$3.4459 \cdot 10^{-8}$	$1.0977 \cdot 10^{-8}$	$3.5491 \cdot 10^{-8}$
L_2	$7.4694 \cdot 10^{-8}$	$2.1244 \cdot 10^{-7}$	$5.7044 \cdot 10^{-8}$	$1.9847 \cdot 10^{-7}$
L_3	$6.4188 \cdot 10^{-8}$	$8.6458 \cdot 10^{-8}$	$6.0665 \cdot 10^{-8}$	$8.9150 \cdot 10^{-8}$

$$\mathbf{\Phi} = \mathbf{\Psi}(T, 0) = [\hat{\boldsymbol{S}}(T)]^T \bar{\boldsymbol{b}} \tag{33}$$

where $\bar{\boldsymbol{b}} = [\boldsymbol{B}_1, \cdots, \boldsymbol{b}_6]$.

Illustrative Example

We shall illustrate the formalism introduced above using the Earth–moon system, for which $e = 0.0549$ and $\mu = 0.01215$. The collinear equilibria of (3) for the Earth–moon system in normalized coordinates are given by

$$L_1 = -1.005062402, \qquad L_2 = 0.8569180073, \qquad L_3 = 1.155679913$$

The constant a_{41}, a_{52}, a_{63} of the linear system (7) are elaborated in Table 1 for the three collinear libration points.

Using Chebyshev polynomials of the first kind of order 15 ($m = 16$), the monodromy matrices for each of the libration points in the Earth-moon system are as follows:

$$\mathbf{\Phi}|_{L_1} = \begin{bmatrix} 2.402 & -0.0033773 & 0 & 0.85357 & 0.92461 & 0 \\ -0.95657 & 0.99635 & 0 & -0.92461 & 0.36146 & 0 \\ 0 & 0 & 0.5358 & 0 & 0 & 0.83983 \\ 2.5494 & -0.0098274 & 0 & 0.55277 & 1.6795 & 0 \\ -2.8015 & -0.0039381 & 0 & -1.7038 & -0.85287 & 0 \\ 0 & 0 & -0.84888 & 0 & 0 & 0.5358 \end{bmatrix} \tag{34}$$

$$\mathbf{\Phi}|_{L_2} = \begin{bmatrix} 11.42 & -1.6914 & 0 & 2.9827 & 1.4593 & 0 \\ -4.6526 & 0.23762 & 0 & -1.4593 & -0.26311 & 0 \\ 0 & 0 & -0.64401 & 0 & 0 & 0.32893 \\ 32.678 & -5.8632 & 0 & 8.5017 & 4.1264 & 0 \\ -16.502 & 0.97735 & 0 & -4.2741 & -2.6809 & 0 \\ 0 & 0 & -1.7793 & 0 & 0 & -0.64401 \end{bmatrix} \tag{35}$$

$$\mathbf{\Phi}|_{L_3} = \begin{bmatrix} 5.9873 & -0.78497 & 0 & 1.7784 & 1.1612 \\ -2.6628 & 0.44309 & 0 & -1.1612 & 0.012505 \\ 0 & 0 & -0.2147 & 0 & 0 \\ 12.806 & -2.4914 & 0 & 3.6649 & 2.6878 \\ -8.597 & -0.067794 & 0 & -2.7718 & -1.8793 \\ 0 & 0 & -1.7612 & 0 & 0 \end{bmatrix} \tag{36}$$

We now compare these semianalytical results to a numerical integration of system (12) using an 8th-order Dorman–Prince Runge–Kutta algorithm [16]. To study the differences between the two methods, define an error matrix $\boldsymbol{E}$ as the difference between the monodromy matrix obtained from the semianalytical method $\mathbf{\Phi}$ and the monodromy matrix resulting from a direct integration $\mathbf{\Phi}_{\mathrm{dir}}$ so that $\boldsymbol{E} = \mathbf{\Phi} - \mathbf{\Phi}_{\mathrm{dir}}$.

Table 2 lists the error matrix norm (maximum singular value) for the collinear libration points using both kinds of Chebyshev polynomials of order 13 ($m = 14$) and 15 ($m = 16$). The absolute tolerance of both the numerical integration of system (12) and the quadrature of Eq. (22) was set to 10^{-9}.

We note that Chebyshev polynomials of the first kind yield a more accurate approximation than Chebyshev polynomials of the second kind, because the quadrature in Eq. (22) can be better approximated using the structure of the Chebyshev polynomials of the first kind.

The comparison of the semianalytical and the purely numerical methods in Table 2 raises the following question: What if the numerical integration and the semianalytical method are both in error? Ideally, a sound validation of a given method's accuracy should be performed relative to a benchmark analytical solution. Although there is no analytical closed-form solution to the linearized ER3BP state-transition matrix, one can calculate a state-transition matrix for the linearized CR3BP case with $e = 0$. In this case, Eqs. (12) assume the linear time-invariant form

$$\frac{d}{dt}\mathbf{\Psi}(t, t_0) = \boldsymbol{A}\mathbf{\Psi}(t, t_0) \qquad \mathbf{\Psi}(t_0, t_0) = \boldsymbol{I} \tag{37}$$

and the state-transition matrix evaluated at $t = t_0 + T$ is simply

$$\mathbf{\Psi}(t_0 + T, t_0) = e^{\boldsymbol{A}T} \tag{38}$$

As before, we compared the calculation of the state-transition matrix using the semianalytical method and the direct numerical integration. Table 3 provides the error matrix norm (maximum singular value) relative to the analytical solution (38) for both kinds of Chebyshev polynomials of order 13 ($m = 14$) and 15 ($m = 16$). The absolute tolerance of the numerical calculations was set to 10^{-9}.

It is seen that both kinds of Chebyshev polynomials result in a relatively accurate approximation of the state-transition matrix.

Finally, we note that the computation time of the monodromy matrix using the semianalytical method is significantly smaller than the computation time required for a direct numerical integration. For example, the computation time required for evaluating the monodromy matrix using both kinds of Chebyshev polynomials of order 15th is always smaller than 0.02 s, whereas the time required for direct numerical integration is always larger than 0.4 s at a given absolute error tolerance of 10^{-9}.

Conclusions

In this work, we developed a computationally efficient semianalytical method for calculating the transition and monodromy matrices for the linearized elliptic restricted three-body problem (ER3BP). We suggested a simple manipulation, based on the theory of orthogonal polynomials, which transforms the problem of integrating a set of nonautonomous differential equations into solving a system of algebraic equations.

The use of Chebyshev polynomials of both kinds was examined;, however, the first kind gave better results, yielding a monodromy matrix that was extremely close to the monodromy matrix resulting from a high-order numerical integration.

The proposed method is essential for trajectory design of future libration-point missions, because it validates calculations of the monodromy matrix, required for the design of stationkeeping maneuvers.

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