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Manifolds and Metrics in the Relative Spacecraft Motion Problem

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This paper establishes a methodology for obtaining the general solution to the spacecraft relative motion problem by utilizing the Cartesian configuration space in conjunction with classical orbital elements. The geometry of the relative motion configuration space is analyzed, and the relative motion invariant manifold is determined. Most importantly, the geometric structure of the relative motion problem is used to derive useful metrics for quantification of the minimum, maximum, and mean distance between spacecraft for commensurable and noncommensurable mean motions. A number of analytic solutions as well as useful examples are provided, illustrating the calculated bounds. A few particular cases that yield simple solutions are given.

Nomenclature

a	=	semimajor axis
E	=	eccentric anomaly
$\mathcal{E}$	=	follower orbit
e	=	eccentricity
$\mathcal{F}$	=	follower perifocal frame
f	=	true anomaly
$\mathcal{I}$	=	inertial frame
i	=	inclination
J_k	=	Bessel function
$\mathcal{L}$	=	leader-fixed frame
M	=	mean anomaly
n	=	mean motion
n_0	=	fundamental frequency
$\boldsymbol{R}$	=	leader position vector
$\mathcal{R}$	=	relative motion invariant manifold
$\boldsymbol{r}$	=	follower position vector
W	=	distance function
α	=	normalized semimajor axis
μ	=	gravitational constant
$\boldsymbol{\rho}$	=	relative position vector
Ω	=	right ascension of the ascending node
ω	=	argument of periapsis
$\boldsymbol{\omega}$	=	angular velocity vector
$\vert\cdot\vert$	=	vector norm
$\Vert\cdot\Vert$	=	signal norm

Superscripts

$'$	=	leader
$*$	=	relative orbital element

I. Introduction

MODELING relative spacecraft motion is of prime importance for formation-flying satellites and distributed spacecraft systems, which constitute a significant share of planned future space missions. Hence, in recent years, there has been a marked renaissance of research in this field.

The most common model of spacecraft relative motion is the Clohessy–Wiltshire (CW) linear model.[1] The CW linear formulation assumes small deviations from a circular reference orbit and uses the initial conditions as the constants of motion. In addition, the CW model is inherently limited to short-term motion, as it was originally developed for rendezvous applications. Recognizing some of the limitations of this approach, various researchers have generalized the CW equations for eccentric reference orbits.[2–4] Many other researchers have modeled relative motion while taking into account orbital perturbations such as oblateness[5–7] and drag[8] and using high-fidelity models including third-body effects.[9]

An important modification of the CW linear solution is the use of orbital elements as constants of the motion instead of the Cartesian initial conditions. This concept, originally suggested by Hill,[10] has been widely used in the analysis of relative spacecraft dynamics.[11,12] This approach facilitates the examination of the orbital perturbations effect on the relative motion via variational equations such as Lagrange's planetary equations or Gauss's variational equations. Moreover, utilizing orbital elements facilitates the derivation of high-order, nonlinear extensions to the CW solution.

There have been few reported efforts to obtain high-order solutions to the relative motion problem.[7,13–15] Karlgaard and Lutze proposed formulating the relative motion in spherical coordinates in order to derive second-order expressions.[13] The use of canonical orbital elements known as epicyclic elements for modeling relative motion equations has also been proposed.[14,15]

However, there are still a few open questions in the field of relative motion modeling. For instance, there have been little reported efforts to understand the relative spacecraft motion geometry and topology in the general case of two spacecraft flying on arbitrary elliptic orbits. More importantly, analytic, closed-form expressions for metrics quantifying the relative motion space are absent. These important metrics are the minimum, maximum, and mean distance between spacecraft that follow Keplerian elliptic orbits. Calculation of these metrics is enabling for all future formation-flying missions. Knowledge of the minimum distance is essential for collision avoidance, maximum distance is crucial for sensor and control design, and the mean distance is important for power management and line-of-sight calculations.

This paper, therefore, has two main contributions. First, we shall establish a methodology for obtaining the general solution of the spacecraft relative motion problem by utilizing the Cartesian configuration space in conjunction with classical orbital elements. In other words, we are utilizing the known inertial expressions describing vehicles flying on elliptic orbits in order to obtain a closed-form solution in a rotating frame, without resorting to approximations or

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series expansions. We then study the geometry of the relative motion configuration space and find the relative motion invariant manifold. This manifold represents a well-defined region of the configuration space on which the relative motion can evolve. Assuming Keplerian relative motion, all relative motions, be they commensurable (periodic) or noncommensurable (quasi-periodic), will evolve on this invariant manifold. This finding is significant for understanding the nature of relative motion and for studying the relative distance between the spacecraft.

Next, we use the geometric structure of the relative motion problem in order to define and derive useful metrics for quantification of minimum, maximum, and mean distance between spacecraft for both commensurable and noncommensurable mean motions. As the motion is confined to lie on the invariant manifold, relative distances can be easily defined and studied. We provide a number of analytic solutions as well as useful examples for the metrics of interest, illustrating the newly found expressions for the relative motion metrics. We also give a few particular cases that yield simple solutions which should be of interest to the designers of distributed space systems.

II. Relative Motion Modeling

In the following discussion, we shall study the relative motion between two arbitrary elliptic Keplerian orbits. The reference orbit will represent the orbit of a *leader* spacecraft, whereas the additional orbit will represent an orbit of a *follower* spacecraft. To that end, we shall utilize standard coordinate systems as briefly discussed herein.

To begin, consider a Keplerian motion about a primary gravitational body with a center of mass at O. The follower's perifocal frame $\mathcal{F}$ is a Cartesian, dextral coordinate system centered at O defined by the unit vectors $\boldsymbol{f}_1$,$\boldsymbol{f}_2$, constituting the fundamental orbital plane, and $\boldsymbol{f}_3 = \boldsymbol{f}_1 \times \boldsymbol{f}_2$. Equivalently, the leader perifocal frame is $\mathcal{F}'$. For modeling relative motion, it is sometimes beneficial to project the follower's position onto a rotating, local-level local-horizon Euler–Hill frame $\mathcal{L}$, centered at the leader spacecraft. The fundamental plane is the leader orbital plane, defined by the unit vectors $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, and $\boldsymbol{l}_3 = \boldsymbol{l}_1 \times \boldsymbol{l}_2$. The inertial frame of reference $\mathcal{I}$ is a Cartesian, dextral frame defined by the unit vectors $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, constituting the fundamental plane, coinciding with the primary's equatorial plane, and $\boldsymbol{s}_3 = \boldsymbol{s}_1 \times \boldsymbol{s}_2$.

The inertial equations of the leader's motion are given by the standard Newtonian relationship

$$\ddot{\boldsymbol{R}} = -(\mu/R^3)\boldsymbol{R} \tag{1}$$

where

$$R = \|\boldsymbol{R}\| = \frac{a'(1-e'^2)}{(1+e'\cos f')} \tag{2}$$

and a', e', f' are the leader's orbit semimajor axis, eccentricity, and true anomaly, respectively. In a similar fashion, the follower inertial equations of motion are

$$\ddot{\boldsymbol{r}} = -(\mu/r^3)\boldsymbol{r} \tag{3}$$

where

$$r = \|\boldsymbol{r}\| = \frac{a(1-e^2)}{(1+e\cos f)} \tag{4}$$

and a, e, f are the follower's orbit semimajor axis, eccentricity, and true anomaly, respectively. Let

$$[\rho]_{\mathcal{I}} = \boldsymbol{r} - \boldsymbol{R} \tag{5}$$

denote the position of the follower relative to the leader calculated in the inertial frame. Subtracting Eq. (1) from Eq. (3) yields

$$[\ddot{\rho}]_{\mathcal{I}} = -\mu/\|\boldsymbol{R}+\boldsymbol{\rho}\|^3 + (\mu/R^3)\boldsymbol{R} \tag{6}$$

To express the relative acceleration in frame $\mathcal{L}$, we recall that

$$[\ddot{\rho}]_{\mathcal{I}} = \frac{\mathrm{d}^2\boldsymbol{\rho}}{\mathrm{d}t^2} + 2\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}} \times \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} + \frac{\mathrm{d}\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}}}{\mathrm{d}t} \times \boldsymbol{\rho} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}} \times ({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}} \times \boldsymbol{\rho}) \tag{7}$$

where $\boldsymbol{\rho}$ is the relative position in frame $\mathcal{L}$, ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}}$ denotes the angular velocity vector of frame $\mathcal{L}$ relative to frame $\mathcal{I}$, and the operator d(·) denoted differentiation with respect to $\mathcal{L}$.

As ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}}$ is normal to the orbital plane, we can write

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}} = [0, 0, \dot{f}']^T \tag{8}$$

where in the case of Keplerian motion,

$$\begin{aligned}\dot{f}' &= \sqrt{\mu/a'^3(1-e'^2)^3}(1+e'\cos f')^2 \\ &= \left[n'\big/(1-e')^{\frac{3}{2}}\right](1+e'\cos f')^2\end{aligned} \tag{9}$$

and n' is the mean motion of the leader. The position vector of the leader spacecraft in $\mathcal{L}$ can be written as

$$[\boldsymbol{R}]_{\mathcal{L}} = [R, 0, 0]^T \tag{10}$$

Taking

$$\boldsymbol{\rho} = [x, y, z]^T \tag{11}$$

and substituting Eqs. (6), (8), and (11) into Eq. (7), yields the following component-wise well-known differential equations for spacecraft relative motion:

$$\ddot{x} - 2\dot{f}'\dot{y} - \ddot{f}'y - \dot{f}'^2x = -\frac{\mu(R+x)}{[(R+x)^2+y^2+z^2]^{\frac{3}{2}}} + \frac{\mu}{R^2} \tag{12}$$

$$\ddot{y} + 2\dot{f}'\dot{x} + \ddot{f}'x - \dot{f}'^2y = -\frac{\mu y}{[(R+x)^2+y^2+z^2]^{\frac{3}{2}}} \tag{13}$$

$$\ddot{z} = -\frac{\mu z}{[(R+x)^2+y^2+z^2]^{\frac{3}{2}}} \tag{14}$$

Equations (12–14) constitute a six-dimensional system of nonlinear differential equations, incorporating the known Keplerian solutions for R and $\dot{f}'$ [cf. Eqs. (2) and (9), respectively] and the relationship

$$\ddot{f}' = -2\dot{R}\dot{f}'/R$$

These equations admit a single equilibrium at $x = y = z = 0$, meaning that the follower spacecraft will appear stationary in the leader frame if and only if their positions coincide on a given elliptic orbit. This single relative equilibrium bifurcates into an equilibria continuum if the leader is assumed to follow a circular reference orbit. (If the dynamics are transformed into a rotating-pulsating Euler–Hill frame such that time dependence is mapped into true anomaly dependence, the equilibria of the circular problem are recovered in the elliptic problem.)

The configuration space for the relative spacecraft dynamics of Eqs. (12–14) is $\mathbb{R}^3$. Let $T_S(\mathbb{R}^3) = \mathbb{R}^3 \times \mathbb{R}^3$ be the tangent space of $\mathbb{R}^3$. We shall use $(\boldsymbol{\rho}, \dot{\boldsymbol{\rho}})$ as coordinates for $T_S(\mathbb{R}^3)$, that is to say, $(\boldsymbol{\rho}, \dot{\boldsymbol{\rho}}) \in T_S(\mathbb{R}^3)$.

The nonlinear relative equations of motion can be straightforwardly solved (in terms of true anomaly) because the generating orbits are Keplerian. To see this, we shall find an expression for $\boldsymbol{\rho}$ using consecutive Eulerian rotations and a translation.

The initial step is to transform from $\mathcal{F}$ to $\mathcal{I}$ using three consecutive clockwise rotations conforming to the common $3-1-3$ sequence. To that end, we define the line of nodes (LON) obtained from the intersection of the follower's orbital plane and the inertial reference plane.

The composite rotation, $T \in SO(3)$, transforming any vector in $\mathcal{F}$ into the inertial frame $\mathcal{I}$ is given by[16,17]

$$T(\omega, i, \Omega) = \begin{bmatrix} c_\Omega c_\omega - s_\Omega s_\omega c_i & -c_\Omega s_\omega - s_\Omega c_\omega c_i & s_\Omega s_i \\ s_\Omega c_\omega + c_\Omega s_\omega c_i & -s_\Omega s_\omega + c_\Omega c_\omega c_i & -c_\Omega s_i \\ s_\omega s_i & c_\omega s_i & c_i \end{bmatrix} \tag{15}$$

where i, Ω, ω are the follower's inclination, right ascension of the ascending node (RAAN), and argument of periapsis, respectively, and we have used the compact notation $s_x = \sin(x)$, $c_x = \cos(x)$. The next step is to transform from $\mathcal{I}$ to the leader's perifocal frame $\mathcal{F}'$, using the rotation matrix $T^T(\omega', i', \Omega')$, where i', Ω', ω' are the leader's inclination, RAAN, and argument of periapsis, respectively. The transformation of the follower's position vector from $\mathcal{F}'$ into $\mathcal{L}$ requires an additional rotation,

$$T_1(f') = \begin{bmatrix} c_{f'} & s_{f'} \\ -s_{f'} & c_{f'} \end{bmatrix} \tag{16}$$

and a translation by $[\boldsymbol{R}]_{\mathcal{L}}$, resulting in

$$\boldsymbol{\rho} = T_1(f')T^T(\omega', i', \Omega')T(\omega, i, \Omega)[\boldsymbol{r}]_{\mathcal{F}} - [\boldsymbol{R}]_{\mathcal{L}} \tag{17}$$

where $[\boldsymbol{r}]_{\mathcal{F}}$ is the follower's position vector in frame $\mathcal{F}$, expressed in terms of the follower's eccentric anomaly E as

$$[\boldsymbol{r}]_{\mathcal{F}} = \begin{bmatrix} a(\cos E - e) \\ b \sin E \\ 0 \end{bmatrix} \tag{18}$$

and $b = a\sqrt{(1-e^2)}$.

Equation (17), written component-wise, is the most general solution (which still entails solution of Kepler's equation, as explained in the sequel) to the relative motion problem, modeled by the differential equations (12–14). This solution can be simplified if we utilize relative orbital elements. These orbital elements describe the orientation of the follower's orbital plane relative to the leader's orbital plane, where the relative LON is defined by the intersection of these two planes. Using this LON and some fixed reference line, we can define the relative RAAN Ω^*, the relative argument of periapsis ω^*, and the relative inclination i^*. In terms of the relative elements, the expression for the relative position is simplified into

$$\boldsymbol{\rho} = T_1(f')T(\omega^*, i^*, \Omega^*)[\boldsymbol{r}]_{\mathcal{F}} - [\boldsymbol{R}]_{\mathcal{L}} \tag{19}$$

Substituting Eqs. (10), (15), (16), and (18) into Eq. (19) yields

$$x = \tfrac{1}{2}[(k_3 - k_2)s_{f'-E} + (k_1 + k_4)c_{f'-E} + (k_3 + k_2)s_{f'+E} + (k_1 - k_4)c_{f'+E}] - e(k_3 s_{f'} + k_1 c_{f'}) - R \tag{20}$$

$$y = \tfrac{1}{2}[-(k_1 + k_4)s_{f'-E} + (k_3 - k_2)c_{f'-E} + (k_4 - k_1)s_{f'+E} + (k_2 + k_3)c_{f'+E}] + e(k_1 s_{f'} - k_3 c_{f'}) \tag{21}$$

$$z = k_5(c_E - e) + k_6 s_E \tag{22}$$

where

$$k_1 = (c_{\Omega^*}c_{\omega^*} - s_{\Omega^*}s_{\omega^*}c_{i^*})a \tag{23}$$

$$k_2 = (-c_{\Omega^*}s_{\omega^*} - s_{\Omega^*}c_{\omega^*}c_{i^*})b \tag{24}$$

$$k_3 = (s_{\Omega^*}c_{\omega^*} + c_{\Omega^*}s_{\omega^*}c_{i^*})a \tag{25}$$

$$k_4 = (-s_{\Omega^*}s_{\omega^*} + c_{\Omega^*}c_{\omega^*}c_{i^*})b \tag{26}$$

$$k_5 = s_{\omega^*}s_{i^*}a \tag{27}$$

$$k_6 = c_{\omega^*}s_{i^*}b \tag{28}$$

We can simplify Eqs. (20–22) by adopting the magnitude-phase representation

$$x = K_1 \sin(f' - E + \Phi_1) + K_2 \sin(f' + E + \Phi_2) - K_3 \sin(f' + \Phi_3) - R \tag{29}$$

$$y = K_1 \sin(f' - E - \Phi_1) + K_2 \sin(f' + E - \Phi_2) + K_3 \sin(f' - \Phi_3) \tag{30}$$

$$z = K_4 \sin(E + \Phi_4) - k_5 e \tag{31}$$

where

$$K_1 = \tfrac{1}{2}\sqrt{(k_3 - k_2)^2 + (k_1 + k_4)^2} \tag{32}$$

$$K_2 = \tfrac{1}{2}\sqrt{(k_3 + k_2)^2 + (k_1 - k_4)^2} \tag{33}$$

$$K_3 = e\sqrt{k_1^2 + k_3^2} \tag{34}$$

$$K_4 = \sqrt{k_5^2 + k_6^2} \tag{35}$$

$$\Phi_1 = \tan^{-1}[(k_3 - k_2)/(k_1 + k_4)] \tag{36}$$

$$\Phi_2 = \tan^{-1}[(k_3 + k_2)/(k_1 - k_4)] \tag{37}$$

$$\Phi_3 = \tan^{-1}(k_1/k_3) \tag{38}$$

$$\Phi_4 = \tan^{-1}(k_6/k_5) \tag{39}$$

Thus, we have obtained the general solution for the nonlinear differential equations (12–14) modeling the relative motion problem. This general solution lies in the three-dimensional configuration space, comprising the relative motion invariant manifold $\mathcal{R}$. In this context, $\mathcal{R}$ constitutes a manifold in a loose sense, that is, any solution of $\boldsymbol{f}(x, y, z) = 0$ yielding a two-dimensional manifold in $\mathbb{R}^3$ (i.e., $\mathcal{R}$ is not a topological space). This manifold is invariant because any solution of the relative motion problem starting on $\mathcal{R}$ will remain on $\mathcal{R}$ for all times.

The dynamics on $\mathcal{R}$ evolves according to Eq. (9) and a similar relationship that holds for the follower's eccentric anomaly, emanating from Kepler's equation,

$$\dot{E} = n/(1 - e\cos E) \tag{40}$$

where n is the mean motion of the follower. Thus, the general solution is a function of the leader orbital elements œ′ and the follower orbital elements œ. The orbital elements themselves constitute a manifold in the parameter space, that is,

$$\text{œ} = [a, e, i, \Omega, \omega, M_0]^T \in \Gamma \subset \mathbb{R}^2 \times \mathbb{S}^4 \tag{41}$$

where Γ is an open subset of $\mathbb{R}^2$ and $\mathbb{S}^4$ is the four-sphere.

All possible solutions of the relative motion problem will evolve on $\mathcal{R}$, hence its invariance property. If the mean motions of the leader and follower commensurate (e.g., in a 1:1 resonance, i.e., $n = n'$), then the relative orbit will be a closed smooth curve $\gamma_c(t) \in \mathcal{R}$ satisfying the periodicity condition $\gamma_c(t) = \gamma_c(t + T)$. Otherwise, an open curve $\gamma_o(t) \in \mathcal{R}$ will be obtained, and the motion will be quasi-periodic. Because the dynamics are always confined to evolve on $\mathcal{R}$, the relative motion will be always bounded. (This observation is trivial because the relative motion analyzed here is Keplerian. Nevertheless, many of current works dealing with relative motion tend to distinguish between "bounded" and "unbounded" relative motion, while implicitly referring to 1:1 commensurable and noncommensurable motions, respectively.) We shall use this property in the next section to calculate distances.

Considering the relative motion invariant manifold, a few issues of practical interest to formation flying and distributed spacecraft systems should be analyzed, such as converting $\mathcal{R}$ into a Riemannian manifold by defining some metric and the geodesics on $\mathcal{R}$. We shall be primarily interested in investigating metrics defined on $\mathcal{R}$ for purposes of evaluating the minimum, maximum, and mean distance between spacecraft.

Interestingly, in some instances, the manifold $\mathcal{R}$ can be approximated by parametric representations of familiar geometric shapes. For example, when Ω^* and ω^* are first-order small, the relative position components x, y, z constitute the parametric equations of an elliptic torus (a surface of revolution that is a generalization of the ring torus, which is produced by rotating an ellipse in the xz plane about the z axis), meaning that it can be also written in the form

$$x \approx (c_1 + c_2 \cos v)\cos u \tag{42}$$

$$y \approx (c_1 + c_2 \cos v)\sin u \tag{43}$$

$$z \approx c_3 \sin v \tag{44}$$

where $u, v \in [0, 2\pi]$.

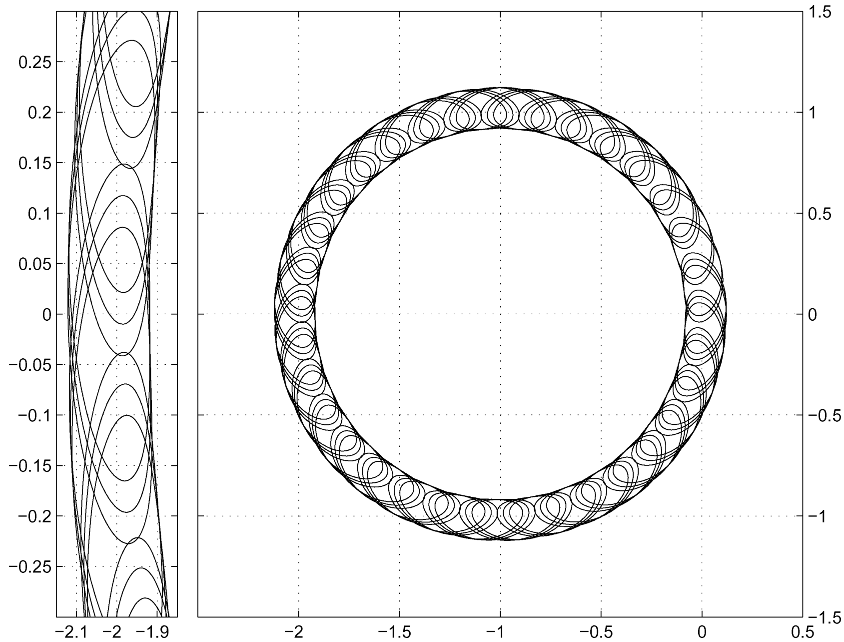

Fig. 1 Quasi-periodic motion of a follower spacecraft in a leader-fixed rotating reference frame. The right pane shows the unit epicycle. The left pane presents a magnified view of an epicycle segment.

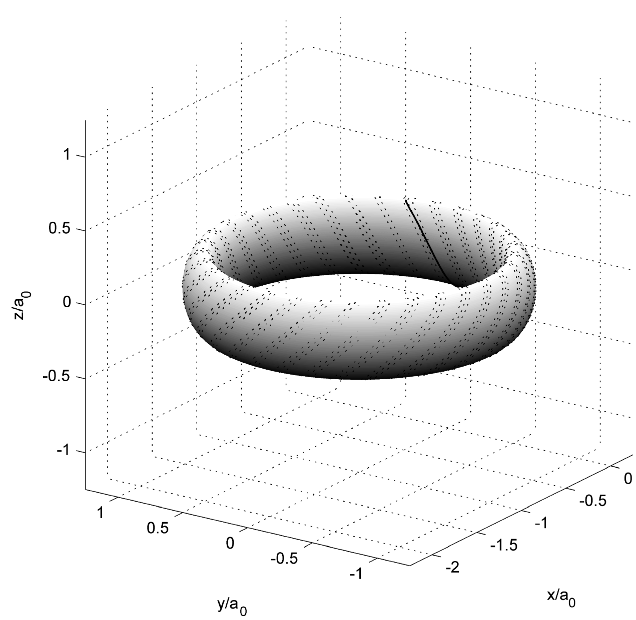

Fig. 2 Commensurable (——) and epicyclic (· · · ·) relative motion on an elliptic torus.

Example 1: Consider an equatorial leader orbit with $a' = 6578$ km and a follower orbit with $a = 6710$ km, $e = 0.1$, $i = 15$ deg, $\Omega = 5$ deg, and $\omega = M_0 = 0$.

To delineate the relative motion geometry, we need to integrate Eqs. (9) and (40). We then substitute the parameter values and the time histories of ν and E into Eqs. (29–31) and plot the results. Figure 1 shows the xy projection of the relative motion using the normalization $a' = n' = \mu = 1$. As $a' \neq a$, the energy/period matching condition is violated, and so we do not expect a 1:1 periodic motion. Instead, we see that the motion is quasi-periodic. The follower spacecraft performs an epicyclic motion along the unit circle in the leader-fixed rotating frame. Thus, although the relative motion is not 1:1 commensurable, it is certainly bounded, as relative motion between elliptic Keplerian orbits will always remain bounded regardless of any particular selection of coordinates or resonance conditions.

Figure 2 shows the three-dimensional motion and the torus that it lies upon (dotted line). The motion clearly evolves along the elliptic torus. If we change the semimajor axis of the follower spacecraft to match that of the leader, the drift will stop, and a closed relative orbit will result (solid line). The closed orbit lies, again, on the three-dimensional elliptic torus.

III. Distances in the Unperturbed Relative Motion Problem

As just mentioned, the relative motion in the elliptic case is always bounded. Let us find the quantitative characteristics of distances between two spacecraft moving along the relative motion invariant manifold $\mathcal{R}$.

Denote by $Q(E)$ a point lying on the Keplerian ellipse representing the leader's orbit $\mathcal{E}$, having the eccentric anomaly E. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be the unit vectors directed toward $Q(0)$ and $Q(\pi/2)$ respectively, and $\boldsymbol{S} = \sqrt{(1-e^2)}\boldsymbol{Q}$. Note that $\boldsymbol{P}$ is the first and $\boldsymbol{Q}$ is the second vector columns of the matrix $T(\omega, i, \Omega)$ [Eq. (15)]. Equivalently, we let $\mathcal{E}'$ be the Keplerian elliptic orbit of the follower.

Let $\rho = |\boldsymbol{\rho}|$ denote the Euclidean vector norm of the relative position vector. Reference 18 used the fact that both orbits are Keplerain in order to find the relative position vector. After some simplification, it was shown[18] that the distance $\rho(E, E')$ between points Q and Q' can be determined by the formula

$$\begin{aligned} W(E, E') \triangleq \rho^2/2aa' &= W_0 + W_1 \cos E + W_2 \sin E + W_3 \cos E' \\ &+ W_4 \sin E' + 2(W_5 \cos E \cos E' + W_6 \cos E \sin E' \\ &+ W_7 \sin E \cos E' + W_8 \sin E \sin E') \\ &+ W_9 \cos 2E + W_{10} \cos 2E' \end{aligned} \tag{45}$$

Here,

$$4W_0 = 2(\alpha + \alpha') + \alpha e^2 + \alpha' e'^2 - 4PP'ee', \qquad W_1 = PP'e' - \alpha e$$

$$W_2 = P'Se', \qquad W_3 = PP'e - \alpha' e', \qquad W_4 = PS'e$$

$$2W_5 = -PP', \qquad 2W_6 = -PS'$$

$$2W_7 = -P'S, \qquad 2W_8 = -SS'$$

$$4W_9 = \alpha e^2, \qquad 4W_{10} = \alpha' e'^2$$

PP', PS', $P'S$, and SS' are scalar products of the corresponding vectors, and

$$\alpha = a/a', \qquad \alpha' = a'/a$$

For the analysis of relative spacecraft motion, it is important to calculate the maximal distance $\rho_{\max}$ for determining the maximum required power of intersatellite communication, choosing the most appropriate sensor for a given maximal range and deriving a suitable relative motion controller. For the purpose of collision avoidance, the minimal distance between satellites $\rho_{\min}$ is needed. In addition, it is required to get an estimate of the mean-squared distance

$$\|\rho\| = \lim_{T\to\infty}\sqrt{\frac{1}{T}\int_0^\infty \rho^2[E(t), E'(t)]\,\mathrm{d}t} \tag{46}$$

so as to determine the average power of the satellite communication system.

A. Mean-Squared Distance

Consider firstly the nonresonant case. Then the motion is quasi-periodic in the sense of Levitan[19] and Bohr[20] with two basic frequencies n, n', which are incommensurable real numbers. As just shown, the relative motion will take place on the relative motion invariant manifold. Hence, averaging with respect to time and with respect to a corresponding two-dimensional torus coincide,[19,20] and Eq. (46) becomes

$$\|\rho\|^2 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \rho^2[E(M), E'(M')]\,\mathrm{d}M\,\mathrm{d}M'$$

where M is the mean anomaly. Transforming to eccentric anomalies yields

$$\|\rho\|^2 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \rho^2(E, E') \times (1 - e\cos E)(1 - e'\cos E')\,\mathrm{d}E\,\mathrm{d}E'$$

Using Eq. (45), we finally obtain

$$\|\rho\|^2 = aa'(2W_0 - eW_1 - e'W_3 + ee'W_5) \tag{47}$$

Remark 1: If e, e' are small quantities of the same order of magnitude, then Eq. (47) implies

$$\|\rho\|^2 = a^2 + a'^2 + \mathcal{O}(e^2) \tag{48}$$

Neglecting $\mathcal{O}(e^2)$, the mean distance is equal to $\sqrt{(a^2 + a'^2)}$ independent of the mutual inclination.

Consider now a resonant case

$$n = mn_0, \qquad n' = m'n_0 \tag{49}$$

with relatively prime natural m, m'. Then

$$M = m\tau, \qquad M' = m'\tau + M_0' - (m'/m)M_0 \tag{50}$$

with

$$\tau = n_0 t + M_0/m \tag{51}$$

The function ρ is 2π periodic with respect to τ, so that Eq. (46) becomes

$$\|\rho\|^2 = \frac{1}{2\pi}\int_0^{2\pi}\rho^2[E(\tau), E'(\tau)]\,\mathrm{d}\tau \tag{52}$$

By taking into account Eq. (45), we can calculate the integral term by term using the absolutely and uniformly convergent series (in the domain $0 \le e \le 1$, $-\infty \le M \le \infty$) (Ref. 21)

$$\cos E = \sum_{k=0}^{\infty} c_k(e)\cos kM, \qquad \sin E = \sum_{k=1}^{\infty} s_k(e)\sin kM \tag{53}$$

where

$$c_0 = -e/2, \qquad c_k = (2/k)J_k'(ke)$$
$$s_k = (2/ke)J_k(ke), \qquad k \ge 1$$

J_k being Bessel functions; the mean value of $\cos 2E$ with respect to M vanishes. Consider the product

$$2\cos E\cos E' = \sum_{k,k'=0}^{\infty} c_k(e)c_{k'}(e')(\cos\varphi_{k,k'} + \cos\varphi_{k,-k'})$$

with

$$\varphi_{k,k'} = (km + k'm')\tau + k'[M_0' - (m'/m)M_0]$$

The mean value of $\cos\varphi_{k,k'}$ with respect to τ vanishes except for the case $k = k' = 0$. The mean value of $\cos\varphi_{k,-k'}$ vanishes except for $km - k'm' = 0$, which is possible if and only if $k = m'k''$ and $k' = mk''$, where k'' is a nonnegative integer. Similar statements are valid for terms of the form $\cos E\sin E'$. Finally, we get

$$\|\rho\|^2/aa' = 2W_0 - eW_1 - e'W_3 + W_5(ee' + 2B_1) + 2(W_6B_2 + W_7B_3 + W_8B_4) \tag{54}$$

where

$$B_1 = \sum_{k=1}^{\infty} c_{m'k}(e)c_{mk}(e')\cos k(mM_0' - m'M_0)$$

$$B_2 = -\sum_{k=1}^{\infty} c_{m'k}(e)s_{mk}(e')\sin k(mM_0' - m'M_0)$$

$$B_3 = \sum_{k=1}^{\infty} s_{m'k}(e)c_{mk}(e')\sin k(mM_0' - m'M_0)$$

$$B_4 = \sum_{k=1}^{\infty} s_{m'k}(e)s_{mk}(e')\cos k(mM_0' - m'M_0)$$

Remark 2: If e, e' are small quantities of the same order of magnitude, then

$$c_k, s_k = \mathcal{O}\left(e^{|k-1|}\right) \Longrightarrow B_i = \mathcal{O}(e^{m+m'-2})$$

Hence, if $m = m' = 1$ the value (54) differs essentially from the nonresonant one (47) even if $e = e' = 0$. If $m = 1, m' = 2$ or $m = 2, m' = 1$, then they differ by the quantity of the order of e. For $m + m' \ge 4$ the formula (48) holds true.

B. Maximal and Minimal Distances

In the nonresonant case

$$\inf_t \rho[E(t), E'(t)] = \min_{E,E'} \rho(E, E')$$
$$\sup_t \rho[E(t), E'(t)] = \max_{E,E'} \rho(E, E') \tag{55}$$

so the problem is reduced into solving the equations

$$\frac{\partial W(E, E')}{\partial E} = 0, \qquad \frac{\partial W(E, E')}{\partial E'} = 0$$

or, equivalently,

$$A\sin E' + B\cos E' = C$$
$$F\sin E' + N\cos E' = K\sin E'\cos E' \tag{56}$$

Here,

$$A = PS' \sin E - SS' \cos E, \qquad B = PP' \sin E - P'S \cos E$$

$$C = e'B - \alpha e \sin E(1 - e \cos E)$$

$$F = PP' \cos E + P'S \sin E + \alpha' e' - PP'e$$

$$N = PS'e - SS' \sin E - PS' \cos E, \qquad K = \alpha' e'^2 \tag{57}$$

are trigonometric polynomials in E of degree 0, 1, or 2.

The optimal algorithm to solve Eq. (56) can be found in Ref. 22. In this case, it is sufficient to solve the single-variable equation

$$g(E) = 0 \tag{58}$$

where g is a trigonometric polynomial of degree 8,

$$g(E) = K^2(A^2 - C^2)(B^2 - C^2)$$
$$+ 2KC[NA(A^2 - C^2) + FB(B^2 - C^2)] - (A^2 + B^2)$$
$$\times [N^2(A^2 - C^2) + F^2(B^2 - C^2) - 2NFAB] \tag{59}$$

After solving Eq. (58), we obtain from the first of the Eqs. (56)

$$\cos E' = \frac{BC + mA\sqrt{D}}{A^2 + B^2}, \qquad \sin E' = \frac{AC - mB\sqrt{D}}{A^2 + B^2} \tag{60}$$

with

$$D = A^2 + B^2 - C^2, \qquad m = \pm 1 \tag{61}$$

The sign of m should be chosen so as to satisfy the second of Eqs. (56).

In degenerate cases, g can be replaced by trigonometric polynomials of degree smaller than 8.

Example 2: Maximal and minimal distance with circular leader and follower orbits.

In this case, g can be replaced by a trigonometric polynomial g_1 of degree 1:

$$g_1 = BF - AN = W_{11} \sin 2E - W_{12} \cos 2E$$

with

$$2W_{11} = (PP')^2 + (PS')^2 - (P'S)^2 - (SS')^2$$

$$W_{12} = (PP')(P'S) + (PS')(SS')$$

The roots of g_1 can be straightforwardly calculated:

$$\tan 2E = W_{12}/W_{11}, \qquad \cot 2E = W_{11}/W_{12}$$

Choosing a reference frame such that $\boldsymbol{P} = \boldsymbol{P}' = (1, 0, 0)$, $\boldsymbol{Q} = (0, 1, 0)$, $\boldsymbol{Q}' = (0, \cos i^*, k \sin i^*)$, i^* being the relative inclination and $k = \pm 1$, we get $W_{12} = 0$. Extremal distances correspond to position of both points on the mutual LON, so that

$$\max \rho = a' + a, \qquad \min \rho = |a' - a| \tag{62}$$

Example 3: Coplanar leader and follower orbits with a circular leader orbit ($\mathcal{E}'$).

Letting $\boldsymbol{P} = \boldsymbol{P}' = (1, 0, 0)$ and $\boldsymbol{Q} = \boldsymbol{Q}' = (0, 1, 0)$ we obtain

$$A = -\sqrt{1 - e^2} \cos E, \qquad M = \cos E - e$$

$$B = \sin E, \qquad N = -\sqrt{1 - e^2} \sin E$$

$$C = -\alpha e \sin E\,(1 - e \cos E), \qquad K = 0.$$

and

$$g_2(E) = \sin^2 E[1 - \alpha^2(1 - e \cos E)^2]$$

This case also leads to an elementary solution. The two roots are lying on the line of apsides,

$$E_1 = 0, \qquad E_2 = \pi$$

If $|\alpha - 1| > \alpha e$, other real roots are absent. If

$$|\alpha - 1| \le \alpha e \Leftrightarrow a(1 - e) \le a' \le a(1 + e) \tag{63}$$

there are two additional roots

$$E_{3,4} = \pm \arccos[(\alpha - 1)/\alpha e]$$

Evidently, these values correspond to intersection points. Hence, if the condition (63) is not fulfilled,

$$\max \rho = a' + a(1 + e)$$

$$\min \rho = \min[|a' - a(1 - e)|, |a' - a(1 + e)|] \tag{64}$$

If condition (63) is satisfied, then

$$\max \rho = a' + a(1 + e), \qquad \min \rho = 0 \tag{65}$$

Consider now the resonant case (49) and (50). The functions W and ρ [compare to Eq. (45)] are 2π periodic with respect to τ [compare to Eq. (51)]. The problem is therefore reduced to determining all roots $\tau_k \in [0, 2\pi)$ of the single-variable function $\tilde{g}(\tau)$, given by

$$\tilde{g} = \frac{\partial W(E, E')}{\partial E} \frac{m}{1 - e \cos E} + \frac{\partial W(E, E')}{\partial E'} \frac{m'}{1 - e' \cos E'} \tag{66}$$

Instead of $\tilde{g}$, we can use a third-order trigonometric polynomial in E and E', denoted by g^*, given by

$$g^* = m(1 - e' \cos E') \frac{\partial W(E, E')}{\partial E} + m'(1 - e \cos E) \frac{\partial W(E, E')}{\partial E'} \tag{67}$$

Note that neither $\tilde{g}(\tau)$ nor $g^*(\tau)$ are trigonometric polynomials in τ.

To calculate $\tilde{g}(\tau)$ or $g^*(\tau)$ for any τ, it is necessary to solve two Kepler equations for determining E and E'.

Example 4: Consider the orbits of Example 2 in the resonant case.

In this case,

$$W = (\alpha + \alpha')/2 - \cos M \cos M' - \cos i \sin M \sin M'$$

$$g^* = m(\sin M \cos M' - \cos i \cos M \sin M')$$
$$+ m'(\cos M \sin M' - \cos i \sin M \cos M')$$

with M, M' linearly dependent on τ [Eq. (50)]. The functions W, g^* are trigonometric polynomials in τ of degree $m + m'$.

Example 5: Consider the orbits of Example 4 in case $m = m' = 1$.

In this case,

$$g^* = (1 - \cos i) \sin(M + M') = (1 - \cos i) \sin(2\tau + M_0' - M_0)$$

If $i = 0$, then $g^* \equiv 0$, and $W =$ const, yielding

$$\rho \equiv 2a|\sin[(M_0' - M_0)/2]|$$

If $i > 0$, then g^* has four roots on the unit circle, so that

$$\min \rho^2 = a^2(1 + \cos i)[1 - \cos(M_0' - M_0)]$$

$$\max \rho^2 = a^2[3 - \cos i - (1 + \cos i) \cos(M_0' - M_0)]$$

which constitute a particularly simple expression for evaluating the extremal intervehicle distances.

IV. Conclusions

We have managed to obtain general expressions for modeling the relative spacecraft geometry and have explicitly parameterized the relative motion configuration space using classical orbital elements as constants of the unperturbed Keplerian motion. Based on this geometric insight, we presented analytic expressions for a few metrics that are important for generating safe and reliable spacecraft formations. We conclude that these expressions differ in the commensurable and incommensurable cases. In some instances, it is possible to simplify the resulting expressions using a few relieving assumptions regarding the orbital eccentricity.

We analyzed a few examples of practical interest to the design of distributed space systems and formation flying. A few important conclusions can be drawn from these examples. First, we conclude that the relative motion geometry evolves on an invariant manifold, which can be easily characterized using the relative orbital elements. The motion along this manifold is quasi-periodic in the general case and periodic in the commensurable case. Both types of motion evolve on the relative motion manifold.

Second, we conclude that if the eccentricities of the leader and follower orbits are first-order small, then the mean-square distance between the orbits is a geometric mean of the leader and follower semimajor axes.

Third, it can be concluded that the extremal distances in the commensurable case are simple functions of the mean anomaly at epoch of the follower and leader spacecraft.

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