Celestial Mechanics and Dynamical Astronomy (2005) 92:337–370
DOI 10.1007/s10569-004-6441-7 

# CANONICAL MODELLING OF RELATIVE SPACECRAFT MOTION VIA EPICYCLIC ORBITAL ELEMENTS

N. JEREMY KASDIN[1], PINI GURFIL[2], and EGEMEN KOLEMEN[1]
[1]*Mechanical and Aerospace Engineering Department, Princeton University, Princeton, NJ 08544, USA, e-mails: jkasdin@princeton.edu, ekolemen@princeton.edu*
[2]*Faculty of Aerospace Engineering, Technion – Israel Institute of Technology, Haifa 32000, Israel, e-mail: pgurfil@technion.ac.il*

(Received: 1 August 2003; accepted: 18 November 2004)

**Abstract.** This paper presents a Hamiltonian approach to modelling spacecraft motion relative to a circular reference orbit based on a derivation of canonical coordinates for the relative state-space dynamics. The Hamiltonian formulation facilitates the modelling of high-order terms and orbital perturbations within the context of the Clohessy–Wiltshire solution. First, the Hamiltonian is partitioned into a linear term and a high-order term. The Hamilton–Jacobi equations are solved for the linear part by separation, and new constants for the relative motions are obtained, called epicyclic elements. The influence of higher order terms and perturbations, such as Earth's oblateness, are incorporated into the analysis by a variation of parameters procedure. As an example, closed-form solutions for $J_2$-invariant orbits are obtained.

**Key words:** Hamiltonian dynamics, relative motion, perturbations, formation flying

## 1. Introduction

The analysis of relative spacecraft motion constitutes an issue of increasing interest due to existing and planned spacecraft formation flying and orbital rendezvous missions. It was in the early 1960s that Clohessy and Wiltshire first published their celebrated work that utilized a Hill-like rotating Cartesian coordinate system to derive expressions for the relative motion between satellites in the context of a rendezvous problem (Clohessy and Wiltshire, 1960). The Clohessy–Wiltshire (CW) linear formulation assumed small deviations from a circular reference orbit and used the initial conditions as the constants of the unperturbed motion. Since then, recognizing the limitations of this approach, others have generalized the CW equations for eccentric reference orbits (Carter and Humi, 1987; Inalhan et al., 2002), and to include perturbed dynamics (Alfriend and Schaub, 2000; Gim and Alfriend, 2001; Scheeres et al., 2003).

An important modification of the CW linear solution is the use of orbital elements as constants of motion instead of the Cartesian initial conditions. This concept, originally suggested by Hill (1878), has been widely used both in the analysis of relative spacecraft motion (Schaub et al., 2000) and in dynamical astronomy (Namouni, 1999). Using this approach allows the effects of orbital perturbations on the relative motion to be examined via variational equations such as Lagrange's planetary equations (LPEs) or Gauss's variational equations (GVEs). Moreover, utilizing orbital elements facilitates the derivation of high-order, nonlinear extensions to the CW solution (Gurfil and Kasdin, 2004).

There have been a few reported efforts to obtain high-order solutions to the relative motion problem. Recently, Karlgaard and Lutze (2001) proposed formulating the relative motion in spherical coordinates in order to derive second-order expressions. The use of Delaunay elements has also been proposed. For instance, Alfriend and Yan (2002) derived differential equations in order to incorporate perturbations and high-order nonlinear effects into the modelling of relative dynamics.

The CW equations, obtained by utilizing Cartesian coordinates to model the relative motion state-space dynamics, usually cannot be solved in closed-form for arbitrary generalized perturbing forces; on the other hand, the orbital elements or Delaunay-based representations can be straightforwardly expanded to treat orbital perturbations, but they utilize characteristics of the inertial, absolute orbits. Hence, using orbital elements or Delaunay variables constitutes an indirect representation of the relative motion problem.

This paper unifies the merits of the CW and the orbital elements-based approaches by developing a Hamiltonian methodology that models the relative motion dynamics using canonical coordinates. The procedure, via solution of the Hamilton–Jacobi equation, is identical to that leading to the classical Delaunay variables, except that it is performed to first order in the rotating Hill frame. The Hamiltonian formulation facilitates the modelling of high-order terms and orbital perturbations via variation of parameters while allowing us to obtain closed form solutions for the relative motion.

We start by deriving the Lagrangian for motion relative to a circular orbit in Cartesian coordinates. Then, using a Legendre transformation, we calculate the Hamiltonian for the relative motion. We partition the Hamiltonian into a linear term and a high-order term. We then solve the Hamilton–Jacobi (HJ) equation for the linear part by separation, obtaining new constants for the relative motion which we call epicyclic elements. These elements can then be used to define the parameters of a relative motion orbit or, more importantly, they can be used to predict the effect of perturbations via variation of parameters.

As an example, we study the effect of $J_2$ induced perturbations on a relative motion orbit. We also show how the canonical approach can be used to find general $J_2$-invariant relative orbits similar to those in Schaub and Alfriend (Schaub and Alfriend, 2001).

## 2. The Lagrangian

The most convenient coordinate system for this problem is the one in which the Hamilton–Jacobi equation most easily separates. We also would like to operate in a coordinate system that most directly allows us to utilize control and simulation techniques. Cartesian coordinates turn out to be most convenient on all counts. Most of the work in this paper will be confined to a rotating Cartesian Euler–Hill system as shown in Figure 1. This coordinate system, denoted by $\mathscr{R}$, is defined by the unit vectors $\hat{x}, \hat{y}, \hat{z}$. The origin of this coordinate system is set on a circular reference orbit of radius $a$ about the Earth. It is rotating with mean motion $n = \sqrt{\frac{\mu}{a^3}}$, where $\mu$ is the gravitational constant. The reference orbit plane is the fundamental plane, the positive $\hat{x}$-axis points radially outward, the $\hat{y}$-axis is rotated 90° in the direction of motion on the reference orbit and lies in the fundamental plane, and the $\hat{z}$-axis completes the setup to yield a Cartesian dextral system.

For simplicity, we treat the case of a relative motion with respect to a circular reference orbit. This is the most common problem and should easily reduce to the Clohessy–Wiltshire (CW) equations. We start with this case because of its simplicity, allowing us to focus attention on the details of the method. Nevertheless, we find that the resulting canonical perturbation equations still provide new and meaningful results. In future work we will present the more involved case of arbitrary elliptical orbits.

The first step is to develop the Lagrangian of relative motion in the rotating frame $\mathscr{R}$. The velocity of the follower spacecraft in $\mathscr{R}$ is given by the usual equation:

$$\mathbf{v} = {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{R}} \times \mathbf{r}_1 + \frac{\mathrm{d}^{\mathscr{R}}}{\mathrm{d}t}\boldsymbol{\rho} + {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{R}} \times \boldsymbol{\rho} \tag{1}$$

where $\mathbf{r}_1 \in \mathbb{R}^3$ is the inertial position vector of the leader spacecraft along the reference orbit, $\rho = [x, y, z]^{\mathrm{T}} \in \mathbb{R}^3$ is the relative position vector in the rotating frame, and ${}^{\mathscr{I}}\omega^{\mathscr{R}} = [0, 0, n]^{\mathrm{T}}$ is the angular velocity of the rotating frame $\mathscr{R}$ with respect to the inertial frame $\mathscr{I}$. Assuming a circular reference

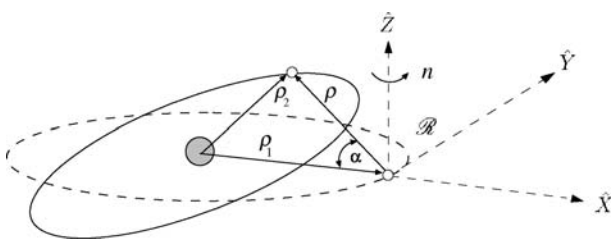


*Figure 1.* Relative motion rotating Euler–Hill reference frame.

orbit, denoting $\|\mathbf{r}_1\| = a$ and substituting into Equation (1), we can write the velocity in $\mathscr{R}$ in a component-wise notation:

$$\mathbf{v} = \begin{bmatrix} \dot{x} - ny \\ \dot{y} + nx + na \\ \dot{z} \end{bmatrix} \tag{2}$$

The kinetic energy per unit mass is given by

$$\mathcal{K} = \frac{1}{2}\|\mathbf{v}\|^2 \tag{3}$$

The potential energy (for a spherical attracting body) of the follower spacecraft, whose position vector is $\mathbf{r_2}$, is the usual gravitational potential written in terms of $\rho = \|\boldsymbol{\rho}\|$ and expanded using Legendre polynomials:

$$\begin{aligned} \mathcal{U} &= -\frac{\mu}{\|\mathbf{r}_2\|} = -\frac{\mu}{\|\mathbf{r}_1 + \boldsymbol{\rho}\|} \\ &= -\frac{\mu}{a\left[1 + 2\frac{\mathbf{r}_1\cdot\boldsymbol{\rho}}{a^2} + \left(\frac{\rho}{a}\right)^2\right]^{1/2}} \\ &= -\frac{\mu}{a}\sum_{k=0}^{\infty} P_k(\cos\alpha)\left(\frac{\rho}{a}\right)^k \end{aligned} \tag{4}$$

where the $P_k(\cos\alpha)$ are the Legendre polynomials,[1]

$$\cos\alpha = -\frac{\boldsymbol{\rho}\cdot\mathbf{r}_1}{a\rho} = \frac{-x}{\sqrt{x^2 + y^2 + z^2}} \tag{5}$$

and $\alpha$ is the angle between the reference orbit radius vector and the relative position vector, as shown in Figure 1.

The Lagrangian $\mathcal{L} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^1$ is now easily found by subtracting the potential energy from the kinetic energy,

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}\left\{(\dot{x} - ny)^2 + (\dot{y} + nx + na)^2 + \dot{z}^2\right\} \\ &+ n^2a^2\sum_{k=0}^{\infty} P_k(\cos\alpha)\left(\frac{\rho}{a}\right)^k \end{aligned} \tag{6}$$

As in the treatment leading to the Clohessy–Wiltshire equations for relative motion, we examine only small deviations from the reference orbit. Thus, we only consider the first three terms of the potential energy,

$$\mathcal{U}^{(0)} = -\frac{\mu}{a} - \frac{\mu}{a^2}\rho\cos\alpha - \frac{\mu}{a^3}\rho^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) \tag{7}$$

and then use Equation (5) to find the low order Lagrangian,

[1]This expansion results from use of the well known generating function $g(t,x) = (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^{n-1}$.

$$\mathcal{L}^{(0)} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + n(x\dot{y} - y\dot{x} + a\dot{y}) + \frac{3}{2}n^2a^2 + \frac{3}{2}n^2x^2 - \frac{n^2}{2}z^2 \tag{8}$$

with the perturbed part of the Lagrangian equal to the higher order terms in the potential $[\mathcal{O}((\rho/a)^3)]$. As a check, it is useful to derive the linear relative equations of motion via the Euler–Lagrange equations on the Lagrangian in Equation (8). Omitting the details, it is straightforward to derive the usual CW equations:

$$\ddot{x} - 2n\dot{y} - 3n^2x = Q_x \tag{9}$$

$$\ddot{y} + 2n\dot{x} = Q_y \tag{10}$$

$$\ddot{z} + n^2z = Q_z \tag{11}$$

where $(Q_x, Q_y, Q_z)$ are the generalized forces in the relative motion frame. For the analysis that follows, we set $Q_x = Q_y = Q_z = 0$.

It is helpful before proceeding further to normalize our equations and simplify the notation. Normalizing rates by $n$ (so time is in units of radians, or, equivalently, the argument of latitude, $u$) and relative distances by $a$ (so all distances are fractions of the reference orbit radius), the normalized Lagrangian is given by:

$$\bar{\mathcal{L}}^{(0)} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + ((x+1)\dot{y} - y\dot{x}) + \frac{3}{2} + \frac{3}{2}x^2 - \frac{1}{2}z^2 \tag{12}$$

where now the dot over a variable represents differentiation with respect to normalized time ($u$) and the coordinates $(x, y, z)$ are dimensionless (and small).

It is also straightforward to change coordinates, writing the Lagrangian in the new coordinate system, and then use the Euler–Lagrange equations to find the equations of motion in a new coordinate system. For example, in Kasdin and Gurfil (2004) we derive the equations of motion in cylindrical coordinates.

## 3. The Hamilton–Jacobi Solution

The overall objective is to divide the three-degree-of-freedom Hamiltonian $\mathcal{H} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^1$ into a linear part and a perturbed part,

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$$

and then solve the Hamilton–Jacobi equation for the unperturbed, linear system. This solution will provide us with new canonical coordinates and momenta that are constants of the (relative) motion. The perturbation, or variation of parameters, equations will then show how these constants vary under various disturbances or higher order terms contained in $\mathcal{H}^{(1)}$.

Finding the Hamiltonian for the Cartesian system is straightforward. First, the canonical momenta are found from the usual definition:

$$
\begin{aligned}
p_x &= \frac{\partial \mathcal{L}^{(0)}}{\partial \dot{x}} = \dot{x} - y \\
p_y &= \frac{\partial \mathcal{L}^{(0)}}{\partial \dot{y}} = \dot{y} + x + 1 \\
p_z &= \frac{\partial \mathcal{L}^{(0)}}{\partial \dot{z}} = \dot{z}
\end{aligned}
\tag{13}
$$

and then, using the Legendre transformation $\mathcal{H} = \dot{q}_i p_i - \mathcal{L}$, the unperturbed Hamiltonian for relative motion in cartesian coordinates is found:

$$
\mathcal{H}^{(0)} = \frac{1}{2}(p_x + y)^2 + \frac{1}{2}(p_y - x - 1)^2 + \frac{1}{2}p_z^2 - \frac{3}{2} - \frac{3}{2}x^2 + \frac{1}{2}z^2 \tag{14}
$$

This Hamiltonian is used to solve the Hamilton–Jacobi equation (see Appendix A). Because the Hamiltonian is a constant, Hamilton's principal function easily separates into a time dependent part summed with Hamilton's characteristic function,

$$
S(x, y, z, u) = W(x, y, z) - \alpha_1'' u
$$

where $\alpha_1''$ is the constant value of the unperturbed Hamiltonian, $\mathcal{H}^{(0)}$. The Hamilton–Jacobi equation then reduces to a partial differential equation in $W(x, y, z)$:

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\partial W}{\partial x} + y\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial y} - x - 1\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial z}\right)^2 \\
&\quad - \frac{3}{2} - \frac{3}{2}x^2 + \frac{1}{2}z^2 = \alpha_1''
\end{aligned}
\tag{15}
$$

Not unexpectedly, the $z$-coordinate easily separates. Separating the characteristic function as $W(x, y, z) = W'(x, y) + W_3(z)$, the HJ equation separates into

$$
\alpha_2' = \frac{1}{2}\left(\frac{\mathrm{d}W_3}{dz}\right)^2 + \frac{1}{2}z^2 \tag{16}
$$

$$
\begin{aligned}
\alpha_1'' + \frac{3}{2} - \alpha_2' = &\frac{1}{2}\left(\frac{\partial W'}{\partial x} + y\right)^2 \\
&+ \frac{1}{2}\left(\frac{\partial W'}{\partial y} - x - 1\right)^2 - \frac{3}{2}x^2
\end{aligned}
\tag{17}
$$

where $\alpha_2'$ has been added and subtracted from Equation (15) for convenience.

Equation (16) is just the HJ equation for simple harmonic motion (which we expect from the well known solution of the CW equations).

It is easily solved via quadrature:

$$W_3(z) = \int \sqrt{2\alpha_2' - z^2}\,\mathrm{d}z = \frac{1}{2}\left[ z\sqrt{2\alpha_2' - z^2} + 2\alpha_2' \sin^{-1}\left(\frac{z}{\sqrt{2\alpha_2'}}\right)\right] \tag{18}$$

The solution of Equation (17) is more subtle. We separate by using the well known in-plane constant of integration of the CW equations. Setting $\alpha_3'$ equal to the integral of Equation (10) with the generalized force equal to zero ($Q_y = 0$) and putting it in terms of the canonical momentum, the third integration constant of the HJ equation is

$$\alpha_3' = p_y + x - 1 \tag{19}$$

Using the fact that $p_y = \partial W'/\partial y$, the remaining HJ equation (17) separates if we let

$$W'(x, y) = W_1(x) + W_2(y) - yx \tag{20}$$

so that

$$\frac{\mathrm{d}W_2}{\mathrm{d}y} = \alpha_3' + 1$$

and thus $W_2 = (\alpha_3' + 1)y$ by quadrature. Equation (17) then simplifies to

$$\left(\frac{\mathrm{d}W_1}{\mathrm{d}x}\right)^2 + x^2 - 4\alpha_3' x = 2\alpha_1' - (\alpha_3')^2 \tag{21}$$

where we have used $\alpha_1' = \alpha_1'' - \alpha_2' + 3/2$. This equation is again easily integrated for $W_1$ by quadrature,

$$W_1 = \int \sqrt{2\alpha_1' - (\alpha_3')^2 + 4\alpha_3' x - x^2}\,\mathrm{d}x \tag{22}$$

yielding

$$W_1 = \frac{-2\alpha_3' + x}{2}\sqrt{2\alpha_1' + 3(\alpha_3')^2 - (-2\alpha_3' + x)^2} - \frac{2\alpha_1' + 3(\alpha_3')^2}{2}\sin^{-1}\left(\frac{2\alpha_3' - x}{\sqrt{2\alpha_1' + 3(\alpha_3')^2}}\right) \tag{23}$$

The final generating function from the solution of the low-order HJ equation is thus given by

$$S(x, y, z, \alpha_1', \alpha_2', \alpha_3', u) = W_1(x) + W_2(y) + W_3(z) - yx - (\alpha_1' + \alpha_2')(u - u_0) \tag{24}$$

(Note that we have omitted the constant 3/2 as it does not affect the solution and, again, we use argument of latitude, $u$, rather than time, introducing an arbitrary initial angle, $u_0$, where we use the usual definition of the zero of $u$ being at the nodal crossing). It is straightforward to express the new canonical momenta $(\alpha_1', \alpha_2', \alpha_3')$ in terms of the original cartesian positions and velocities (and thus in terms of the initial conditions). For instance, $\alpha_3'$ is given by Equation (19) using Equation (13). Equation (16) is used to find $\alpha_2'$, substituting $p_z$ from Equation (13) for $\mathrm{d}W_3/\mathrm{d}z$. Finally, $\alpha_1' = \alpha_1'' - \alpha_2' + 3/2$ is simply the value of the Hamiltonian and is thus given by Equation (14) with the momenta substituted from Eq. (13). The result is

$$\alpha_1' = \frac{1}{2}(p_x + y)^2 + \frac{1}{2}(p_y - x - 1)^2 - \frac{3}{2}x^2 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{3}{2}x^2 \tag{25}$$

$$\alpha_2' = \frac{1}{2}p_z^2 + \frac{1}{2}z^2 = \frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2 \tag{26}$$

$$\alpha_3' = p_y + x - 1 = \dot{y} + 2x \tag{27}$$

The canonical coordinates $(Q_1', Q_2', Q_3')$ or the corresponding constant phase variables $(\beta_1', \beta_2', \beta_3')$ are found via the partial derivatives of the generating functions in Equation (24) with respect to each of the new canonical momenta,

$$Q_i' = \frac{\partial\left[S(x, y, z, \alpha_1', \alpha_2', \alpha_3', u) + \alpha_1'(u - u_0)\right]}{\partial \alpha_i'} = \frac{\partial\left[W(x, y, z, \alpha_1', \alpha_2', \alpha_3')\right]}{\partial \alpha_i'} \tag{28}$$

yielding

$$\begin{aligned} Q_1' &= u - u_0 + \beta_1' = \tan^{-1}\left(\frac{x - 2\alpha_3'}{\sqrt{2\alpha_1' - (\alpha_3')^2 + 4\alpha_3' x - x^2}}\right) \\ &= -\tan^{-1}\left(\frac{3x + 2\dot{y}}{\dot{x}}\right) \end{aligned} \tag{29}$$

$$\begin{aligned} Q_2' &= u - u_0 + \beta_2' = \tan^{-1}\left(\frac{z}{\sqrt{2\alpha_2' - z^2}}\right) \\ &= \tan^{-1}\left(\frac{z}{\dot{z}}\right) \end{aligned} \tag{30}$$

$$\begin{aligned} Q_3' &= \beta_3' = y - 2\sqrt{2\alpha_1' - (\alpha_3')^2 + 4\alpha_3' x - x^2} \\ &\qquad + 3\alpha_3' \tan^{-1}\left(\frac{x - 2\alpha_3'}{\sqrt{2\alpha_1' - (\alpha_3')^2 + 4\alpha_3' x - x^2}}\right) \\ &= -(3\dot{y} + 6x)\tan^{-1}\left(\frac{3x + 2\dot{y}}{\dot{x}}\right) - 2\dot{x} + y \end{aligned} \tag{31}$$

where we have used the definitions of the canonical momenta in terms of velocities in Equations (13) and Equations (25)–(19) to express these in terms of the cartesian positions and rates. Note that the generalized coordinate $Q'_i$ consists of a time term and a constant of the motion (a phase like term), $\beta'_i$.

While these canonical variables can be used in the final equations of motion, one more modification dramatically simplifies the final result. We define a new momentum variable, $\alpha_1 = \alpha'_1 + \frac{3(\alpha'_3)^2}{2}$, and solve for the new low order Hamiltonian:

$$\mathcal{H}^{(0)} = \alpha_1 + \alpha_2 = \alpha'_1 - \frac{3(\alpha'_3)^2}{2} + \alpha'_2 \tag{32}$$

By modifying the generating function accordingly, we obtain equations for the new canonical momenta and coordinates in terms of the Cartesian variables:

$$\begin{aligned}\alpha_1 &= \frac{1}{2}(p_x + y)^2 + 2(p_y - x - 1)^2 + \frac{9}{2}x^2 + 6x(p_y - x - 1)\\ &= \frac{1}{2}\left(\dot{x}^2 + (2\dot{y} + 3x)^2\right)\end{aligned} \tag{33}$$

$$\alpha_2 = \frac{1}{2}p_z^2 + \frac{1}{2}z^2 = \frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2 \tag{34}$$

$$\alpha_3 = p_y + x - 1 = \dot{y} + 2x \tag{35}$$

$$\begin{aligned}Q_1 &= u - u_0 + \beta_1 = \tan^{-1}\left(\frac{x - 2\alpha_3}{\sqrt{2\alpha_1 - 4\alpha_3^2 + 4\alpha_3 x - x^2}}\right)\\ &= -\tan^{-1}\left(\frac{3x + 2\dot{y}}{\dot{x}}\right)\end{aligned} \tag{36}$$

$$\begin{aligned}Q_2 &= u - u_0 + \beta_2 = \tan^{-1}\left(\frac{z}{\sqrt{2\alpha_2 - z^2}}\right)\\ &= \tan^{-1}\left(\frac{z}{\dot{z}}\right)\end{aligned} \tag{37}$$

$$\begin{aligned}Q_3 &= -3\alpha_3(u - u_0) + \beta_3 = y - 2\sqrt{2\alpha_1 - 4\alpha_3^2 + 4\alpha_3 x - x^2}\\ &= -2\dot{x} + y\end{aligned} \tag{38}$$

Solving Equations (33)–(38) for $x, y$, and $z$ yields the *generating solution* for the Cartesian relative position components in terms of the new constants of the motion, the canonical momenta $(\alpha_1, \alpha_2, \alpha_3)$ and the canonical coordinates $(Q_1, Q_2, Q_3)$:

$$x(t) = 2\alpha_3 + \sqrt{2\alpha_1}\sin(Q_1) = 2\alpha_3 + \sqrt{2\alpha_1}\sin(u - u_0 + \beta_1) \tag{39}$$
$$y(t) = Q_3 + 2\sqrt{2\alpha_1}\cos(Q_1) = 3\alpha_3(u - u_0) + \beta_3 + 2\sqrt{2\alpha_1}\cos(u - u_0 + \beta_1) \tag{40}$$
$$z(t) = \sqrt{2\alpha_2}\sin(Q_2) = \sqrt{2\alpha_2}\sin(u - u_0 + \beta_2) \tag{41}$$

From Equations (33)–(38) (or, alternatively, by differentiating Equations (39)–(41) with respect to time) we can also obtain the expressions for the Cartesian relative velocity components in terms of $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$:

$$\dot{x}(t) = \sqrt{2\alpha_1}\cos(u - u_0 + \beta_1) \tag{42}$$

$$\dot{y}(t) = -3\alpha_3 - 2\sqrt{2\alpha_1}\sin(u - u_0 + \beta_1) \tag{43}$$

$$\dot{z}(t) = \sqrt{2\alpha_2}\cos(u - u_0 + \beta_2) \tag{44}$$

Finally, it is often convenient to have expressions for the original canonical momenta in terms of the new elements. These are easily found from the transformation equations:

$$p_x(t) = -Q_3 - \sqrt{2\alpha_1}\cos(u - u_0 + \beta_1) \tag{45}$$

$$p_y(t) = 1 - \alpha_3 - \sqrt{2\alpha_1}\sin(u - u_0 + \beta_1) \tag{46}$$

$$p_z(t) = \sqrt{2\alpha_2}\cos(u - u_0 + \beta_2) \tag{47}$$

These equations are consistent with the well known results from the CW equations:

$$x(t) = \dot{x}_0 \sin u - (2\dot{y}_0 + 3x_0)\cos u + (2\dot{y}_0 + 4x_0) \tag{48}$$

$$y(t) = 2\dot{x}_0 \cos u + (4\dot{y}_0 + 6x_0)\sin u + (y_0 - 2\dot{x}_0) - (3\dot{y}_0 + 6x_0)u \tag{49}$$

$$z(t) = z_0 \cos u + \dot{z}_0 \sin u \tag{50}$$

where rates have again been normalized by the orbit rate, $n$. Thus, in canonical coordinates, the motion consists of a periodic out-of-plane oscillation parameterized by $\alpha_2, \beta_2$, a periodic in-plane motion described by $\alpha_1, \beta_1$, and a secular drift in $y$ given by $\alpha_3$. The usual invariance with $y$ is given by the arbitrary shift $\beta_3$. It is straightforward to show that the generating solution (39)–(41) is identical to the well known solution of the CW equations in terms of initial conditions. We call the new constants of the motion $\Xi = [\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3]$ *epicyclic orbital elements* for the relative motion. They are defined on the manifold $\mathcal{O} \times \mathbb{S}^3$, where $\mathcal{O} = \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \subset \mathbb{R}^3$. For comparison, the Cartesian vectors $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ are defined on the tangent space $\mathbb{R}^3 \times \mathbb{R}^3$.

## 4. Contact Epicyclic Elements

The epicyclic elements described above provide a convenient parameterization of a first-order relative motion orbit in terms of amplitude and phase. This can be particularly convenient when distributing satellites around a periodic orbit (equal amplitudes, but different phases). However, variational equations presented later for these elements can become quite complicated and numerically sensitive. This is particularly a concern when some of the amplitudes approach zero, resulting in the phase terms becoming ill-defined. For these situations it is convenient to introduce an alternative set of constants in terms of only amplitude variables. We call these the *contact epicyclic elements*, label them $\Xi' = [a_1, a_2, a_3, b_1, b_2, b_3]$, and define them via the contact transformation:

$$a_1 = \sqrt{2\alpha_1}\cos\beta_1 \tag{51}$$

$$b_1 = \sqrt{2\alpha_1}\sin\beta_1 \tag{52}$$

$$a_2 = \sqrt{2\alpha_2}\cos\beta_2 \tag{53}$$

$$b_2 = \sqrt{2\alpha_2}\sin\beta_2 \tag{54}$$

$$a_3 = \alpha_3 \tag{55}$$

$$b_3 = \beta_3 \tag{56}$$

It can easily be shown that the transformation in Equations (51)–(56) is symplectic. That is, it is straightforward to show that

$$\left(\frac{\partial \Xi'}{\partial \Xi}\right) J \left(\frac{\partial \Xi'}{\partial \Xi}\right)^{\mathrm{T}} = J \tag{57}$$

where $J$ is the symplecitic matrix $[0, I; -I, 0]$. Thus, the new elements are also canonical and satisfy Hamilton's equations. For some problems, the variational equations for these elements will be easier to work with. In terms of the new canonical momenta $(a_1, a_2, a_3)$ and new canonical coordinates $(b_1, b_2, b_3)$, the Cartesian relative position equations become

$$x(t) = 2a_3 + a_1\sin(u - u_0) + b_1\cos(u - u_0) \tag{58}$$

$$y(t) = b_3 - a_3(u - u_0) - 2b_1\sin(u - u_0) + 2a_1\cos(u - u_0) \tag{59}$$

$$z(t) = b_2\cos(u - u_0) + a_2\sin(u - u_0) \tag{60}$$

Finally, it is important to note that Equations (39)–(44) or Equations (58)–(60) constitute *global* coordinates for the tangent space $\mathbb{R}^3 \times \mathbb{R}^3$ for the relative motion, due to that fact the epicyclic orbital elements are canonical. This means that variations of the parameters $\Xi$ or $\Xi'$ due to perturbations can be obtained via Hamilton's equations on a perturbing Hamiltonian, and the resulting time varying parameters $\Xi(t)$ or $\Xi'(t)$ can then be substituted into the generating solutions (39)–(41) or (58)–(60) to yield the exact relative motion description in the configuration space $\mathbb{R}^3$.

One important caveat is necessary. While nominally either set of elements can be used for perturbation analysis, it is only the original epicyclic elements for which the Hamiltonian splits into a nominal, unperturbed part and a perturbation part. The value of this is that Hamilton's equations can be used on the perturbation Hamiltonian alone (as with Delaunay elements in the two-body problem) in terms of the epicyclic elements. This is not the case for the contact elements, as they do not necessarily solve the H–J equation. The approach we will take is to find the perturbation differential equations for the original epicyclic elements and then perform the transformation to contact elements, using the chain rule to find the desired differential equations for these elements.

## 5. Perturbation Analysis

The primary value of the canonical approach is the ease with which equations for the variations of parameters can be found. Specifically, the variations of the epicyclic orbital elements are given by Hamilton's equations applied to the perturbation Hamiltonian, $\mathcal{H}^{(1)}$:

$$\dot{\alpha}_i = -\frac{\partial \mathcal{H}^{(1)}}{\partial Q_i} \tag{61}$$

$$\dot{\beta}_i = \frac{\partial \mathcal{H}^{(1)}}{\partial \alpha_i} \tag{62}$$

$$\dot{Q}_i = \frac{\partial \mathcal{H}^{(0)}}{\partial \alpha_i} + \dot{\beta}_i \tag{63}$$

These can be used to find the effect on the elements of any number of perturbations which are derived from a conservative potential, such as high-order gravitational harmonics (oblateness) and third-body effects, in a similar manner to the variation of the Delaunay elements (Lagrange's planetary equations can be found via a non-symplectic transformation of the Delaunay elements). To demonstrate the methodology, in the remainder of this section we consider the effect of the Earth's zonal harmonics, particularly the $J_2$ term. For Earth orbiting systems, this is often one of the largest perturbations and the most disruptive for maintaining formations. While all of the results may not be new, this approach is unique, with the benefit of all variables being differential, and this example provides an important verification of the method.

Treating the Earth as axially symmetric, the external potential including the gravitational zonal harmonics, $U_\oplus = \mathcal{U} + U_{\text{zonal}}$, is given by (Battin, 1999):

$$U_{\text{zonal}} = \frac{\mu_\oplus}{\|\mathbf{r_2}\|}\sum_{k=2}^{\infty} J_k \left(\frac{R_\oplus}{\|\mathbf{r_2}\|}\right)^k P_k(\cos\phi) \tag{64}$$

where $\phi$ is the follower spacecraft colatitude angle, given by

$$\cos\phi = \frac{Z}{\|\mathbf{r_2}\|} \tag{65}$$

$Z$ is the normal deflection in an inertial, geocentric–equatorial reference frame and the $J_k$'s are constants, the first three of which are

$$J_2 = 0.00108263 \tag{66}$$

$$J_3 = -0.00000254 \tag{67}$$

$$J_4 = -0.00000161 \tag{68}$$

The perturbed relative motion is then found by setting the perturbing Hamiltonian equal to this potential, $\mathcal{H}^{(1)} = U_{\text{zonal}}$, and substituting $\|\mathbf{r_2}\| = \|\boldsymbol{\rho} + \mathbf{r_1}\|$. By using Equations (39)–(41), the Hamiltonian can be written in terms of the epicyclic elements and then the variational equations of motion can be found via Equations (61)–(63).

### 5.1. EXAMPLE 1: $J_2$ PERTURBATION IN EQUATORIAL ORBITS

To illustrate the analysis, we start with the simpler problem of a circular, equatorial reference orbit and ask for the variational equations of the contact elements describing a relative motion due to $J_2$. Substituting $i = 0$ in the zonal potential, the perturbing Hamiltonian, in terms of the cartesian Euler–Hill coordinates, is given by

$$\mathcal{H}^{(1)} = \frac{n^2 J_2 R_\oplus^2 (2z^2 - 1 - 2x - x^2 - y^2)}{2a^2(1 + 2x + x^2 + y^2 + z^2)^{(5/2)}} \tag{69}$$

where we have again normalized the distances by the reference orbit radius, $a$. Expanding to second order in the elements and substituting for $(x, y, z)$ yields the perturbing Hamiltonian in terms of the epicyclic elements,

$$\begin{aligned}\mathcal{H}^{(1)} = -\frac{1}{2}J_2\left(\frac{R_\oplus}{a}\right)^2 \Big(&1 - 6\alpha_3 - 3\sqrt{2\alpha_1}\sin Q_1 + 24\alpha_3^2 \\ &+24\sqrt{2\alpha_1}\alpha_3\sin Q_1 + 12\alpha_1\sin^2 Q_1 \\ &-\frac{3}{2}Q_3^2 - 6\sqrt{2\alpha_1}Q_3\cos Q_1 \\ &-12\alpha_1\cos^2 Q_1 - 9\alpha_2\sin^2 Q_2\Big)\end{aligned} \tag{70}$$

where we have eliminated the leading $n^2$ term since we again have put time in units of argument of longitude.

Equations (61)–(63) are now used to find the differential equations for the epicyclic elements,

$$\dot{\alpha}_1 = -\frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\begin{pmatrix}-\sqrt{2\alpha_1}\cos Q_1 + 8\sqrt{2\alpha_1}\alpha_3\cos Q_1\\ +2\sqrt{2\alpha_1}Q_3\sin Q_1 + 8\alpha_1\sin 2Q_1\end{pmatrix} \tag{71}$$

$$\dot{\beta}_1 = -\frac{3}{4}\frac{J_2}{\sqrt{\alpha_1}}\left(\frac{R_\oplus}{a}\right)^2\begin{pmatrix}8\sqrt{2}\alpha_3\sin Q_1 - \sqrt{2}\sin Q_1\\ -16\sqrt{\alpha_1}\cos^2 Q_1 - 2\sqrt{2}Q_3\cos Q_1\\ +8\sqrt{\alpha_1}\end{pmatrix} \tag{72}$$

$$\dot{\alpha}_2 = -\frac{9}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\alpha_2\sin 2Q_2 \tag{73}$$

$$\dot{\beta}_2 = \frac{9}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\sin^2 Q_2 \tag{74}$$

$$\dot{\alpha}_3 = -\frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\left(Q_3 + 2\sqrt{2\alpha_1}\cos Q_1\right) \tag{75}$$

$$\dot{Q}_3 = -3\alpha_3 + 3J_2\left(\frac{R_\oplus}{a}\right)^2\left(1 - 8\alpha_3 - 4\sqrt{2\alpha_1}\sin Q_1\right) \tag{76}$$

As alluded to in Section 4, these equations are reasonably complicated and nonlinear, with an evident singularity at $\alpha_1 = 0$. Therefore, it is more convenient, and useful, to change to the contact epicyclic elements, resulting in the time varying differential equations,

$$\dot{a}_1 = \frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\begin{pmatrix}-\cos(u-u_0) + 4\sin(2(u-u_0))a_1\\ +4\cos(2(u-u_0))b_1 + 2\sin(u-u_0)q_3\\ +8\cos(u-u_0)a_3\end{pmatrix} \tag{77}$$

$$\dot{b}_1 = \frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\begin{pmatrix}\sin(u-u_0) + 4\cos(2(u-u_0))a_1\\ -4\sin(2(u-u_0))b_1 + 2\cos(u-u_0)q_3\\ -8\sin(u-u_0)a_3\end{pmatrix} \tag{78}$$

$$\dot{a}_2 = -\frac{9}{4}J_2\left(\frac{R_\oplus}{a}\right)^2((1+\cos(2(u-u_0)))b_2 - \sin(2(u-u_0))a_2) \tag{79}$$

$$\dot{b}_2 = \frac{9}{4}J_2\left(\frac{R_\oplus}{a}\right)^2(\sin(2(u-u_0))b_2 - (1-\cos(2(u-u_0)))a_2) \tag{80}$$

$$\dot{a}_3 = -\frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2(q_3 + 2\cos(u-u_0)a_1 - 2\sin(u-u_0)b_1) \tag{81}$$

$$\dot{q}_3 = -3a_3 + 3J_2\left(\frac{R_\oplus}{a}\right)^2\begin{pmatrix}1 - 8a_3 - 4\sin(u-u_0)a_1\\ -4\cos(u-u_0)b_1\end{pmatrix} \tag{82}$$

where we have used $q_3 = -3a_3(u-u_0) + b_3 = Q_3$

These equations can be used to analyze the effect of $J_2$ on the relative motion in equatorial orbits as well as to search for initial conditions that guarantee bounded formations. As a first check of the validity of the result, we examine the stable, circular orbit solution of a constant radial offset. It is well known that in an equatorial orbit the purely radial perturbed force still allows for a circular orbit but with a modified rate. Alternatively, we can ask for the small, constant radial offset (non-zero $x$) that will produce an orbit with the reference orbit rate. One approach is to simply equate the radial gravitational force with the centrifugal force of an orbit at rate $n$. To first order in $J_2$ the result is an offset of $x_0 = \frac{J_2}{2}\left(\frac{R_\oplus}{a}\right)^2$. This same result can easily be found via the C–W differential equations by inserting a radial perturbing force, $f_r = -\frac{3}{2}\frac{\mu J_2 R_\oplus^2}{r^4}$, setting all rates to zero, and solving for $x_0$ to first order. Again, the result is $x_0 = \frac{J_2}{2}\left(\frac{R_\oplus}{a}\right)^2$. We now search for that same result in Equations (77)–(82). Keeping only terms to first order in $J_2$ and the epicyclic elements, the equations simplify to

$$\dot{a}_1 = -\frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\cos(u-u_0) \tag{83}$$

$$\dot{b}_1 = \frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\sin(u-u_0) \tag{84}$$

$$\dot{a}_2 = 0 \tag{85}$$

$$\dot{b}_2 = 0 \tag{86}$$

$$\dot{a}_3 = 0 \tag{87}$$

$$\dot{q}_3 = -3a_3 + 3J_2\left(\frac{R_\oplus}{a}\right)^2 \tag{88}$$

These equations can be solved by quadrature

$$a_1 = a_1(0) - \frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\sin(u-u_0) \tag{89}$$

$$b_1 = b_1(0) - \frac{3}{2}J_2\left(\frac{R_\oplus}{a}\right)^2\cos(u-u_0) \tag{90}$$

$$a_2 = a_2(0) \tag{91}$$

$$b_2 = b_2(0) \tag{92}$$

$$a_3 = a_3(0) \tag{93}$$

$$q_3 = q_3(0) + 3\left(J_2\left(\frac{R_\oplus}{a}\right)^2 - a_3(0)\right)(u-u_0) \tag{94}$$

One equilibrium solution to these equations has no out-of-plane motion ($a_2(0) = b_2(0) = 0$) and $a_3(0) = J_2\left(\frac{R_\oplus}{a}\right)^2$ to eliminate the drifting term. To see that this is the same radial offset solution, we insert this solution back into Equations (58) and (59) to find that the solution in the Euler–Hill frame is $y = 0$ and $x = \frac{J_2}{2}\left(\frac{R_\oplus}{a}\right)^2$ as expected. This confirms the validity of the approach. We also note that this condition is necessary for any non-drifting solution in the equatorial Euler–Hill frame and it eliminates all effects of $J_2$ to first order.

We also make an observation. One might expect from looking at Equations (58) and (59) that this constant equilibrium would be entirely contained in $a_3$, with $q_3 = a_1 = b_1 = 0$. This is not the case, however. A careful solution of the time varying equations shows that under perturbation, $a_1$ and $b_1$ have an in-phase harmonic solution resulting in the constant result. This is similar to the inertial description of this orbit using Lagrange's planetary equations, which consist of an osculating ellipse of varying eccentricity always tangent to the physical, circular orbit trajectory.

### 5.2. EXAMPLE 2: GENERAL $J_2$ PERTURBATIONS ON RELATIVE MOTION TRAJECTORIES

In our next example, we turn to the full $J_2$ perturbed problem; that is, we allow for inclined reference orbits. The analysis becomes more problematic here. Without even searching for the variational equations, we know that any satellite orbit will have a long term, secular drift in the node angle and argument of perigee induced by oblateness, causing any relative motion analysis to lose validity as the satellite drifts away from the Euler–Hill reference frame. Schaub and Alfriend (Alfriend and Schaub, 2000; Alfriend and Yan, 2002), for example, realizing this, derived general $J_2$-invariant (and almost invariant) satellite formations by matching these drifts among the satellites. In other words, the satellite orbits still drift relative to the usual Hill reference frame, but they drift in such a way that the formation remains bounded.

To find the linearized effects of $J_2$ on a general satellite relative orbit and to search for bounded formations, it is therefore most insightful to return to the original derivation and replace the fixed reference orbit with a circular orbit also rotating at the mean $J_2$ induced drift. The motion can then be examined relative to this drifting reference orbit. We do this is in the following subsections.

#### 5.2.1. *The modified reference orbit and $J_2$ perturbing Lagrangian and Hamiltonian*

As the reference orbit is circular, we need only account for the long-term, uniform drift in the right ascension of the ascending node ($\Omega$) and the long term drift in the argument of latitude ($u = M + \omega$). Allowing the reference orbit to have a drift in the longitude of the ascending node, $\dot{\Omega}$, and argument of latitude, $\dot{u}$, results in a modification to the angular velocity between this modified Hill frame and the inertial frame. Using the 3-1-3 ordered rotation $(\Omega, i, u)$ and the fact that there is no long term drift in inclination, $i$, the angular velocity, expressed in the inertial frame $(\mathscr{I})$, is now given by

$$ {}^{\mathscr{I}}\omega^{\mathscr{R}} = \begin{bmatrix} \dot{\Omega}\sin i \sin u \\ \dot{\Omega}\sin i \cos u \\ \dot{\Omega}\cos i + \dot{u} \end{bmatrix}_{\mathscr{I}} \tag{95} $$

where $u$ is the argument of latitude, $\dot{u} = n + \delta n$ is the modified orbit rate including the $J_2$ perturbation, and $n = \sqrt{\mu/\bar{a}^3}$, $\bar{a}$ being the mean semi-major axis. Note that we are not necessarily making the assumption that there is a satellite on the reference orbit experiencing the $J_2$ perturbations nor are we assuming that these rates represent any physical effect; we are simply modifying the angular velocity of the reference orbit by additional constant rates.

The equation for the rotation rates are somewhat subtle to obtain. The typical expressions are written in terms of the initial or mean semi-major axis of the osculating orbits (see, e.g., Battin, 1999 or Vinti, 1998). Here, however, we select a reference orbit with the radius, $\bar{r}$, equivalent to the mean radius of the $J_2$ perturbed orbit (Born, 2001),

$$ \bar{r} = \bar{a} + \frac{3}{4} J_2 \left( \frac{R_\oplus}{\bar{a}} \right)^2 (3\sin^2 i - 2) \tag{96} $$

Since we are free to select the reference orbit, this equation is solved for $\bar{a}$ and then used to find the mean rates of change of the node angle and argument of latitude (Vinti, 1998; Born, 2001) for the arbitrary, circular reference orbit,

$$ \dot{\Omega} = -\frac{3}{2} \bar{n} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 \cos i \tag{97} $$

$$ \delta n = \frac{3}{4} \bar{n} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 \left( 3 - \frac{7}{2} \sin^2 i \right) \tag{98} $$

where $\bar{n} = \sqrt{\mu/\bar{r}^3}$. Note also that in Equation (98) we have included in $\dot{u}$ both the effect of the rate of change of true anomaly and of the argument of perigee as the reference orbit is circular (i.e., $\dot{u} = \dot{M} + \dot{\omega}$).

We now substitute this angular velocity into Equation (1) to find the satellite's new velocity vector with respect to inertial space:

$$ \mathbf{v} = \begin{bmatrix} \dot{x} - \bar{n}y \\ \dot{y} + \bar{n}x + \bar{n}\bar{r} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} \dot{\Omega}s_i c_u z - \dot{\Omega}c_i y - \delta n y \\ \left\{ \begin{array}{l} (\dot{\Omega}c_i + \delta n)(x + \bar{r}) \\ -\dot{\Omega}s_i s_u z \end{array} \right\} \\ \dot{\Omega}s_i s_u y - \dot{\Omega}s_i c_u (x + \bar{r}) \end{bmatrix} \tag{99} $$

where we have used the notation $s_i = \sin(i), c_i = \cos(i)$, and so on.

The Lagrangian is now computed in the same manner as before by subtracting the potential energy from the kinetic energy, but now including the new velocity expressions in the kinetic energy and the perturbing potential from Equation (64):

$$\mathcal{L} = \frac{1}{2}|\mathbf{v}|^2 + \bar{n}^2\bar{r}^2 \sum_{k=0}^{\infty} P_k(\cos\alpha)\left(\frac{\rho}{\bar{r}}\right)^k - \bar{U}_{\text{zonal}} \tag{100}$$

As we did in the original problem, we simplify by normalizing the Lagrangian. We normalize all distances by the reference orbit radius, $\bar{r}$ and rates (i.e., time) by $\bar{n}$. This results in the normalized Lagrangian:

$$\bar{\mathcal{L}} = \frac{1}{2}|\mathbf{v}^{(0)}|^2 + \mathbf{v}^{(0)} \cdot \mathbf{v}^{(1)} + \frac{1}{2}|\mathbf{v}^{(1)}|^2 + \sum_{k=0}^{\infty} P_k(\cos\alpha)(\rho)^k - \bar{U}_{\text{zonal}} \tag{101}$$

where, as before, all distances are dimensionless, the argument of latitude is used instead of time, and all differentiation is with respect to normalized time. Here, $\mathbf{v}^{(0)}$ is the part of the normalized velocity in the relative motion frame independent of $J_2$ (and the same as the velocity in the original problem),

$$\mathbf{v}^{(0)} = \begin{bmatrix} \dot{x} - y \\ \dot{y} + (x+1) \\ \dot{z} \end{bmatrix} \tag{102}$$

and $\mathbf{v}^{(1)}$ is the small remaining term of order $J_2$,

$$\mathbf{v}^{(1)} = \begin{bmatrix} v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \end{bmatrix} = \begin{bmatrix} \bar{\dot{\Omega}} s_i c_u z - (\bar{\dot{\Omega}} c_i - \bar{\delta n}) y \\ (\bar{\dot{\Omega}} c_i - \bar{\delta n})(x+1) - \bar{\dot{\Omega}} s_i s_u z \\ \bar{\dot{\Omega}} s_i s_u y - \bar{\dot{\Omega}} s_i c_u (x+1) \end{bmatrix} \tag{103}$$

where $\bar{\dot{\Omega}} = \dot{\Omega}/\bar{n}$ and $\bar{\delta n} = \delta n/\bar{n}$.

As before, we keep only the low order terms in the Lagrangian. For this treatment we use only the first order potential as we did earlier and we keep terms only to first order in $J_2$, thus allowing us to immediately drop the second order term, $\frac{1}{2}|\mathbf{v}^{(1)}|^2$ from Equation (101). The result is a low order Lagrangian identical to our earlier treatment (Equation 12) but with two additional perturbing terms:

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}^{(0)} + \mathbf{v}^{(0)} \cdot \mathbf{v}^{(1)} - \bar{U}_{\text{zonal}} \tag{104}$$

We note here that the Euler–Lagrange equations could be used on this Lagrangian to find differential equations for the motion in this new rotating frame, which may have some usefulness for control design, for example.

We are now in a position to study the effect on the relative motion due to the $J_2$ perturbation. Because the perturbation terms in Equation (104) have a velocity dependence, it is not as simple as finding the variations of the epicyclic element equations from before due to the perturbing potential. We must redefine the canonical momenta for this new drifting frame of reference. Using the usual definition,

$$\begin{aligned} p_x &= \frac{\partial \bar{\mathcal{L}}}{\partial \dot{x}} = \dot{x} - y + v_x^{(1)} \\ p_y &= \frac{\partial \bar{\mathcal{L}}}{\partial \dot{y}} = \dot{y} + x + 1 + v_y^{(1)} \\ p_z &= \frac{\partial \bar{\mathcal{L}}}{\partial \dot{z}} = \dot{z} + v_z^{(1)} \end{aligned} \tag{105}$$

and again using the Legendre transformation, $\mathcal{H} = \sum \dot{q}_i p_i - L$, we find the new Hamiltonian,

$$\begin{aligned} \mathcal{H} = &\frac{1}{2}\left(p_x + y - v_x^{(1)}\right)^2 + \frac{1}{2}\left(p_y - (x+1) - v_y^{(1)}\right)^2 + \frac{1}{2}(p_z - v_z)^2 \\ &- \frac{3}{2} - \frac{3}{2}x^2 + \frac{1}{2}z^2 + y v_x^{(1)} - (x+1)v_y^{(1)} + \bar{U}_{\text{zonal}} \end{aligned} \tag{106}$$

There are two possible solution approaches at this stage. The first is to solve the Hamilton–Jacobi equation again but with the low-order Hamiltonian given by the terms in Equation (106) without the perturbing effective potential. This would result in new closed-form equations for the relative motion about the new drifting reference orbit in terms of new, constant $J_2$-dependent epicyclic elements. The variational equations for these elements could then be found via Hamilton's equations on the perturbing effective potential as usual. The advantage of this approach is that the perturbations are entirely velocity (i.e., momentum) independent. Thus, the solution trajectory is guaranteed to be osculating; that is, the solution is tangent everywhere to the physical trajectory that incorporates the variations of the parameters (Efroimsky and Goldreich, 2003, 2004). The disadvantage is that it requires finding a new solution to the H–J equation (a formidable task, particularly since the nominal Hamiltonian is time-varying) and no longer has as clear a connection to the CW solution. We report on this approach in a future paper.

The second approach, and the one we take in the sequel, is to multiply out the terms in Equation (106) that depend upon $\mathbf{v}^{(1)}$ and treat them as perturbations. In other words, we expand the Hamiltonian again into a low order term and perturbing term,

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \tag{107}$$

where $\mathcal{H}^{(0)}$ is the same as in our original problem given by Equation (14) and the perturbing part is given by

$$\mathcal{H}^{(1)} = -p_x v_x^{(1)} - p_y v_y^{(1)} - p_z v_z^{(1)} + \bar{U}_{\text{zonal}} \tag{108}$$

where we have again dropped terms of second order (or higher) in $J_2$.

In this case, the solution to the H–J equation is the same as given previously. The nominal trajectory in the drifting frame is again given by Equations (39)–(41) and the relationship between the epicyclic elements and the canonical momenta and Cartesian position is the same as in Equations (33)–(35). However, the equalities relating the epicyclic momenta to the Cartesian position and velocities (initial conditions) must be modified due to the new definition of the canonical momenta in the drifting frame (Equations (105)). Also, because we have introduced a perturbing Hamiltonian simply due to the rotating frame, the $\alpha_i$ and $\beta_i$ are no longer constant but vary even in the case of no perturbing potential.

The epicyclic elements for this new drifting frame are now given, in terms of normalized positions and velocities, by

$$\alpha_1 = \frac{1}{2}\left(\dot{x} + v_x^{(1)}\right)^2 + \frac{1}{2}(2(\dot{y} + v_y^{(1)}) + 3x)^2 \tag{109}$$

$$\alpha_2 = \frac{1}{2}\left(\dot{z} + v_z^{(1)}\right)^2 + \frac{1}{2}z^2 \tag{110}$$

$$\alpha_3 = \dot{y} + 2x + v_y^{(1)} \tag{111}$$

The phase variables $(\beta_1, \beta_2, \beta_3)$ are found in a similar manner by substituting these expressions for $(\alpha_1, \alpha_2, \alpha_3)$ into Equations (36)–(38). Note also that the well-known constant of the motion in the traditional CW problem $(\alpha_3)$ has been modified to account for the drift of the reference frame due to $J_2$.

The generating solution for the cartesian relative position components in the new drifting frame is the same as before (Equations (39)–(41)),

$$x(t) = 2\alpha_3 + \sqrt{2\alpha_1}\sin(u - u_0 + \beta_1) \tag{112}$$

$$y(t) = -3\alpha_3(u - u_0) + \beta_3 + 2\sqrt{2\alpha_1}\cos(u - u_0 + \beta_1) \tag{113}$$

$$z(t) = \sqrt{2\alpha_2}\sin(u - u_0 + \beta_2) \tag{114}$$

and the expressions for the canonical momenta are also the same,

$$p_x(t) = -Q_3 - \sqrt{2\alpha_1}\cos(u - u_0 + \beta_1) \tag{115}$$

$$p_y(t) = 1 - \alpha_3 - \sqrt{2\alpha_1}\sin(u - u_0 + \beta_1) \tag{116}$$

$$p_z(t) = \sqrt{2\alpha_2}\cos(u - u_0 + \beta_2) \tag{117}$$

However, the cartesian velocities are slightly modified to account for the new drift terms,

$$\dot{x}(t) = \sqrt{2\alpha_1}\cos(u - u_0 + \beta_1) - v_x^{(1)} \tag{118}$$

$$\dot{y}(t) = -3\alpha_3 - 2\sqrt{2\alpha_1}\sin(u - u_0 + \beta_1) - v_y^{(1)} \tag{119}$$

$$\dot{z}(t) = \sqrt{2\alpha_2}\cos(u - u_0 + \beta_2) - v_z^{(1)} \tag{120}$$

Unlike the original problem, it is not straightforward to find the cartesian velocities simply by differentiating the position variables in Equations (112)–(114). Recall that, because the effect of the small drift of the frame is being treated as a perturbation, the epicyclic elements here are not constant, even in the absence of a conservative perturbation potential. Thus, differentiating Equations (112)–(114) requires the inclusion of expressions for the rates of change of the elements as well. Here, we found the velocities from the expressions for the canonical momenta.

Another subtlety in this approach comes in the interpretation of the perturbed motion. The variation of the constants can still be found via Hamilton's equations on $\mathcal{H}^{(1)}$ in Equation (108). However, now the perturbing Hamiltonian is velocity dependent (i.e., a function of the canonical momenta). It is a theorem that for any velocity dependent perturbations, the nominal trajectory is not osculating (Efroimsky and Goldreich, 2003; 2004). In other words, the variational equations we will find for the epicyclic elements can still be used to model the relative motion in the new drifting frame, but the unperturbed elliptic trajectory given by Equations (112)–(114) is not tangent to the perturbed, physical trajectory.

### 5.2.2. *Contact epicyclic elements*

As we did in Section 4 we can convert from the epicyclic elements to the canonical contact set. This results in the same expressions for the cartesian positions,

$$x(t) = 2a_3 + a_1\sin(u - u_0) + b_1\cos(u - u_0) \tag{121}$$

$$y(t) = b_3 - a_3(u - u_0) - 2b_1\sin(u - u_0) + 2a_1\cos(u - u_0) \tag{122}$$

$$z(t) = b_2\cos(u - u_0) + a_2\sin(u - u_0) \tag{123}$$

but, again, a slightly modified version of the cartesian rates,

$$\dot{x}(t) = a_1\cos(u - u_0) - b_1\sin(u - u_0) - v_x^{(1)} \tag{124}$$

$$\dot{y}(t) = -3a_3 - 2a_1\sin(u - u_0) - 2b_1\cos(u - u_0) - v_y^{(1)} \tag{125}$$

$$\dot{z}(t) = a_2\cos(u - u_0) - b_2\sin(u - u_0) - v_z^{(1)} \tag{126}$$

5.2.3. *Variation of parameters*

Finally, we arrive at the derivation of the variational equations for the epicyclic elements relative to the average drifting frame due to $J_2$. First, by substituting for the various expressions in Equation (108) we find the time varying perturbing Hamiltonian (keeping terms only to second order in the cartesian positions and velocities):

$$\mathcal{H}^{(1)} = \frac{J_2}{8}\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} 8 - 24x + 48x^2 - 24y^2 - 24z^2 + 3p_y - 3yp_x + 3xp_y \\ +(12 - 42y^2 - 30z^2 - 36x + 72x^2)\cos^2(u)\cos^2(i) \\ -(12 - 42y^2 - 36x - 30z^2 + 72x^2)\cos^2(u) \\ -\begin{pmatrix} 12 - 42z^2 + 9p_y - 9p_x y + 72x^2 \\ -36x - 30y^2 + 9p_y x \end{pmatrix}\cos^2(i) \\ +(6p_x z - 6p_z x - 6p_z + 12yz)\sin(2i)\cos(u) \\ +(12z - 48xz + 6p_z y - 6p_y z)\sin(2i)\sin(u) \\ -(12y - 48xy)\cos^2(i)\sin(2u) - (48x - 12y)\sin(2u) \end{pmatrix} \tag{127}$$

This perturbing Hamiltonian is used to find the equations of motion for the epicyclic elements via Hamilton's Equations (61)–(63). However, as we described in Section 4, the resulting differential equations are nonlinear in the elements and rather cumbersome to evaluate. Instead, we again convert from the epicyclic elements to the contact elements and use the chain rule to find the resulting linear first-order variational equations. As we did for the equatorial orbit, we drop the homogeneous terms as second-order small and study only the inhomogeneous part of the variational equations due to $J_2$,

$$\dot{a}_1 = -\frac{3}{32}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} \cos(2i - u - u_0) - 7\cos(3u - u_0 + 2i) \\ +14\cos(3u - u_0) + 6\cos(u - u_0 - 2i) \\ +4\cos(u - u_0) - 7\cos(3u - u_0 - 2i) \\ +6\cos(u - u_0 + 2i) - 2\cos(u + u_0) \\ +\cos(u + u_0 + 2i) \end{pmatrix} \tag{128}$$

$$\dot{b}_1 = \frac{3}{32}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} -\sin(u + u_0 + 2i) - \sin(u + u_0 - 2i) \\ +14\sin(3u - u_0) - 7\sin(3u - u_0 + 2i) \\ -7\sin(3u - u_0 - 2i) \\ +6\sin(u - u_0 - 2i) + 6\sin(u - u_0 + 2i) \\ +2\sin(u + u_0) + 4\sin(u - u_0) \end{pmatrix} \tag{129}$$

$$\dot{a}_2 = \frac{3}{8} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 (\cos(2u - u_0 + 2i) - \cos(2u - u_0 - 2i)) \tag{130}$$

$$\dot{b}_2 = -\frac{3}{8} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 (\sin(2u - u_0 + 2i) - \sin(2u - u_0 - 2i)) \tag{131}$$

$$\dot{a}_3 = -\frac{3}{8} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 (\sin(2u - 2i) + \sin(2u + 2i) - 2\sin(2u)) \tag{132}$$

$$\dot{q}_3 = -3a_3 - \frac{9}{16} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 \begin{pmatrix} 4\cos(2u) - 2\cos(2u - 2i) \\ -2\cos(2u + 2i) + 3\cos(2i) + 1 \end{pmatrix} \tag{133}$$

These equations can easily be solved by quadrature. The motion of the satellite in the relative frame is then found by substituting the solutions for the variations of the contact epicyclic elements (in terms of arbitrary initial conditions $(a_1(u_0), b_1(u_0), a_2(u_0), b_2(u_0), a_3(u_0), q_3(u_0))$ into Equations (121)–(123).

As in the equatorial orbit case, we can search for conditions of bounded relative motion. Because of our choice of drifting reference orbit, this is easily accomplished. By examining the equations for $q_3(u)$ (or $\dot{q}_3$), we can solve for the initial condition on $a_3(u_0)$ to eliminate the drift term,

$$a_3(u_0) = \frac{3}{16} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 \begin{bmatrix} 1 + 3\cos(2i) + 2\cos(2u_0) \\ -\cos(2i - 2u_0) - \cos(2i + 2u_0) \end{bmatrix} \tag{134}$$

(Recall that in the unperturbed case, $a_3(u_0) = 0$ was the usual condition in the C–W equations to eliminate the drift term.)

We can substitute this condition on $a_3(u_0)$ into the integrated equations for the elements and then substitute into the cartesian position equations (112)–(114) to find the bounded equations for perturbed motion of $x(u)$, $y(u)$, and $z(u)$,

$$x(u) = a_1(u_0)\sin(u - u_0) + b_1(u_0)\cos(u - u_0) + \frac{1}{32} J_2 \left( \frac{R_\oplus}{\bar{r}} \right)^2 \begin{pmatrix} 4\cos(2u) - 2\cos(2u + 2i) - 2\cos(2u - 2i) \\ +12\cos(u - u_0) + 6\cos(u + u_0) \\ +18\cos(u - u_0 - 2i) + 18\cos(u - u_0 + 2i) \\ -3\cos(u + u_0 - 2i) - 3\cos(u + u_0 + 2i) \\ +14\cos(u - 3u_0) \\ -7\cos(u - 3u_0 + 2i) - 7\cos(u - 3u_0 - 2i) \end{pmatrix} \tag{135}$$

$$y(u) = q_3(u_0) + 2a_1(u_0)\cos(u-u_0) - 2b_1(u_0)\sin(u-u_0) + \frac{1}{32}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} 2\sin(2u) - \sin(2u-2i) - \sin(2u+2i) \\ -24\sin(u-u_0) - 12\sin(u+u_0) - 18\sin(2u_0) \\ +9\sin(2u_0+2i) + 9\sin(2u_0-2i) \\ -36\sin(u-u_0-2i) - 36\sin(u-u_0+2i) \\ +6\sin(u+u_0-2i) + 6\sin(u+u_0+2i) \\ -28\sin(u-3u_0) \\ +14\sin(u-3u_0+2i) + 14\sin(u-3u_0-2i) \end{pmatrix} \tag{136}$$

$$z(u) = a_2(u_0)\sin(u-u_0) + b_2(u_0)\cos(u-u_0) + \frac{3}{16}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2 \begin{pmatrix} \cos(u+2i) - \cos(u-2i) \\ +\cos(u-2u_0+2i) - \cos(u-2u_0-2i) \end{pmatrix} \tag{137}$$

To partially verify these equations, we again examine the case of $i = 0$. By selecting all of the initial conditions to be zero $(a_1(u_0) = a_2(u_0) = b_2(u_0) = q_3(u_0) = 0)$ except for $b_1(u_0) = (3/2)J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2$ and $a_3(u_0) = (3/4)J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2$ (the no-drift condition), the initial cartesian positions and rates are all zero. Substituting into Equations (135)–(137) shows that the satellite remains at the origin of the new, drifting relative motion frame for all time. This is again the known circular orbit equilibrium solution at the equator, where, for a given radius of the orbit, $\bar{r}$, the orbit rate is no longer the Keplerian $\bar{n}$ but instead is given by $\dot{\Omega} + \dot{u} = \bar{n}\left[1 + \frac{3}{4}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2\right]$ as in Section 5.1, where, again, $\bar{n} = \sqrt{\mu/\bar{r}^3}$.[2]

### 5.2.4. *Simulation results*

In this section we present simulations of a few select bounded relative orbit trajectories and compare to a nonlinear simulation of the two-body equations of motion in inertial space incorporating the $J_2$ zonal harmonic as a perturbing force. The nonlinear simulation results are rotated into the relative motion frame for comparison.

The linearized trajectories are found by selecting initial conditions on the epicyclic elements and then substituting into Equations (135)–(137) for

[2]We note again that this modified orbit rate result for circular equatorial orbits with $J_2$ perturbations is only accurate to first-order in $J_2$. The exact expression for the orbit rate is $\bar{n}\left[1 + \frac{3}{2}J_2\left(\frac{R_\oplus}{\bar{r}}\right)^2\right]^{1/2}$. For low Earth orbits the error amounts to tens of meters per orbit.

simulation. In all simulations, $a_3(u_0)$ was selected to satisfy the no-drift condition in Equation (134).

Initial conditions for the nonlinear simulation were found by computing the initial cartesian positions from Equations (112)–(114) at $u = u_0$,

$$x(u_0) = 2a_3(u_0) + b_1(u_0) \tag{138}$$

$$y(u_0) = q_3(u_0) + 2a_1(u_0) \tag{139}$$

$$z(u_0) = b_2(u_0) \tag{140}$$

and the initial rates from Equations (118)–(120),

$$\dot{x}(u_0) = a_1(u_0) - v_x^{(1)}(u_0) \tag{141}$$

$$\dot{y}(u_0) = -3a_3(u_0) - 2b_1(u_0) - v_y^{(1)}(u_0) \tag{142}$$

$$\dot{z}(u_0) = a_2(u_0) - v_z^{(1)}(u_0) \tag{143}$$

These were then rotated and translated into the inertial frame for simulation. All relative orbit simulations are about a nominal, circular reference orbit of 750 km altitude. Except for the no-drift condition on $a_3(u_0)$, the initial conditions for all the other contact epicyclic elements were set to zero (with the exception of the circular, equatorial orbit).

The first simulation in Figure 2 is a nonlinear simulation of the circular equatorial orbit described earlier. Again, the canonical $J_2$ equations show that the vehicle remains at the origin. In this nonlinear simulation, the relative motion has a small radial oscillation of 5 m and a slow, in-track drift of roughly 50 m per orbit. This is due to the neglected $O(J_2^2)$ term in the angular velocity expression of the drifting frame.

Figures 3 and 4 show a small relative motion trajectory about the origin of the $J_2$ drifting, Euler–Hill like frame for a low inclination (28.5°) reference orbit. A $J_2$ invariant formation could be formed by modifying $u_0$ for each of the satellites and thus distributing them about the trajectory. Note the change in the orbit from the elliptical solution of the C–W equations. Also observe how effective the no-drift condition is at forming a bounded orbit even in the nonlinear case. Figure 5 displays the difference between the first-order solution and the exact nonlinear simulation. This error consists of two effects, a very slight drift (almost zero for this inclination) and an oscillatory difference of increasing amplitude, at roughly the orbit rate, that grows to almost 1 km in-track after 5 orbits. This oscillatory error is deceptive, however, as it is primarily due to a difference in the orbit rates of the linearized and nonlinear trajectories (of order $J_2^2$) and thus differencing the trajectories results in a phasing error that grows with each orbit. Alternatively, we looked at how close the two relative orbits were geometrically. Such a

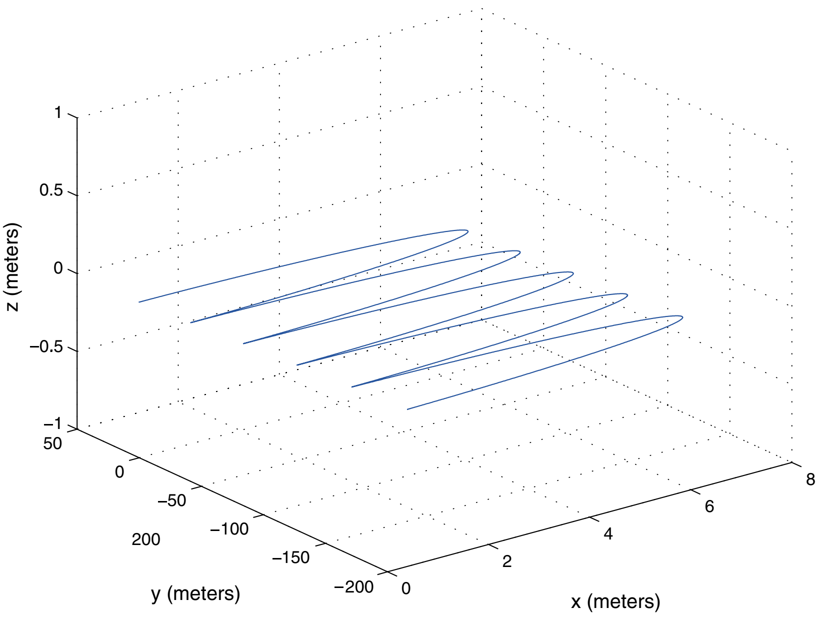


*Figure 2*. Nonlinear simulation of stationary relative motion trajectory in 750 km equatorial reference orbit.

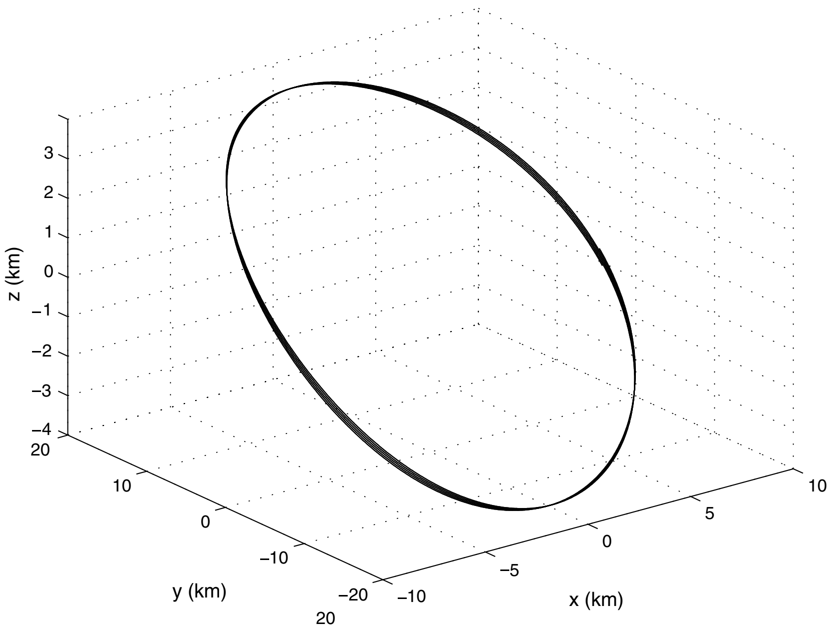


*Figure 3*. Nonlinear simulation of bounded relative motion orbit with $J_2$ Perturbations in 28.5 degree inclination, drifting reference orbit.

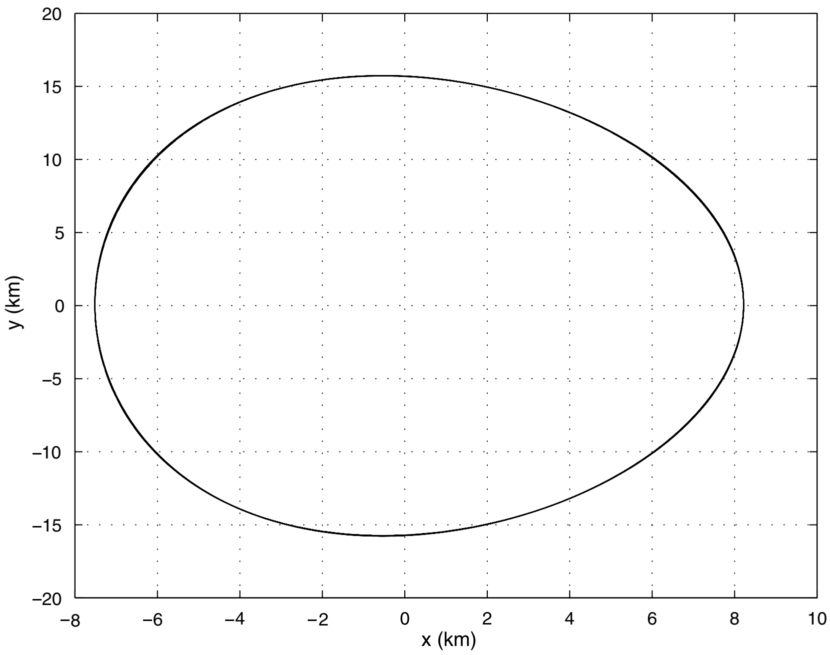


*Figure 4*. *X–Y* projection of relative motion orbit with $J_2$ perturbations in 28.5 degree inclination, drifting reference orbit.

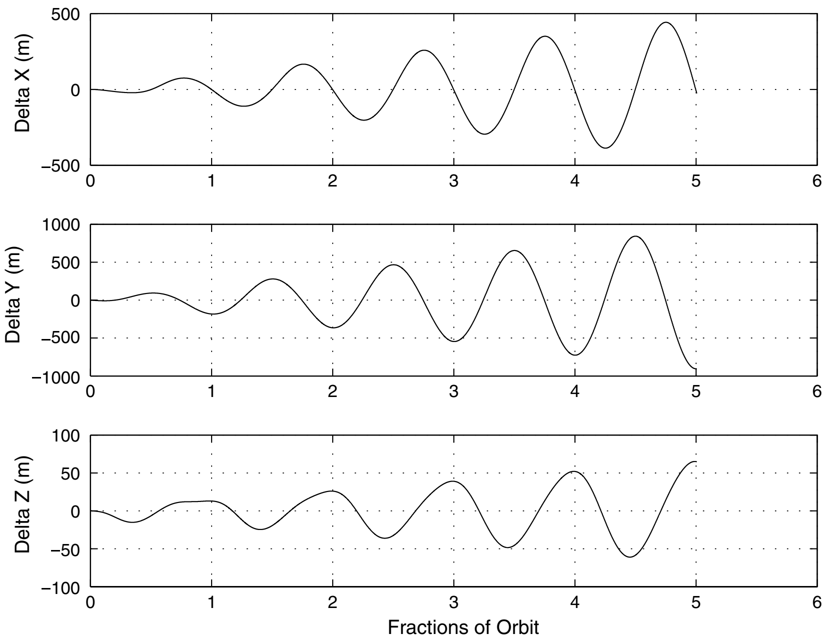


*Figure 5*. A comparison of the *x–y* projection of the relative motion orbit using the linearized, canonical equations and a full, inertial, nonlinear simulation over 5 orbits for a 28.5 degree reference orbit.

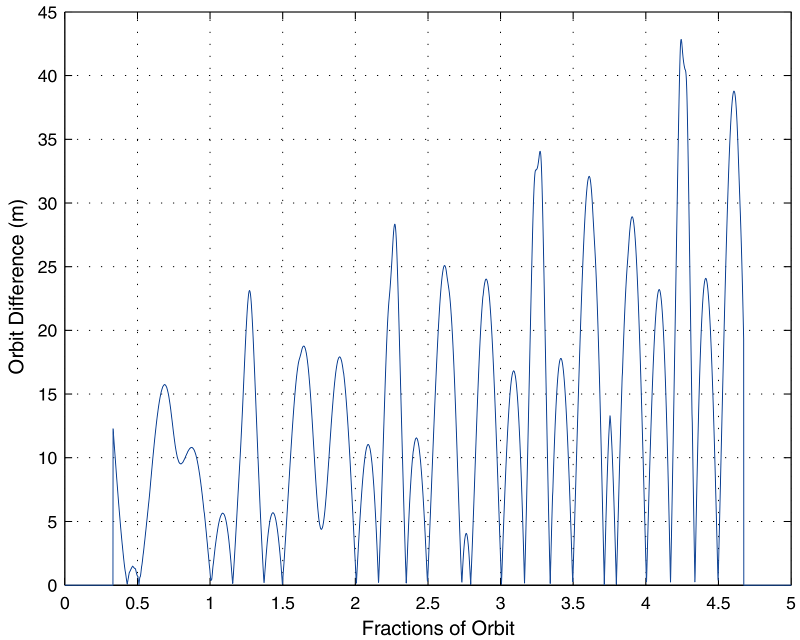


*Figure 6*. A comparison of the relative displacement between the linearized trajectory and that from a full, inertial, nonlinear simulation over 5 orbits for a 28.5 degree reference orbit.

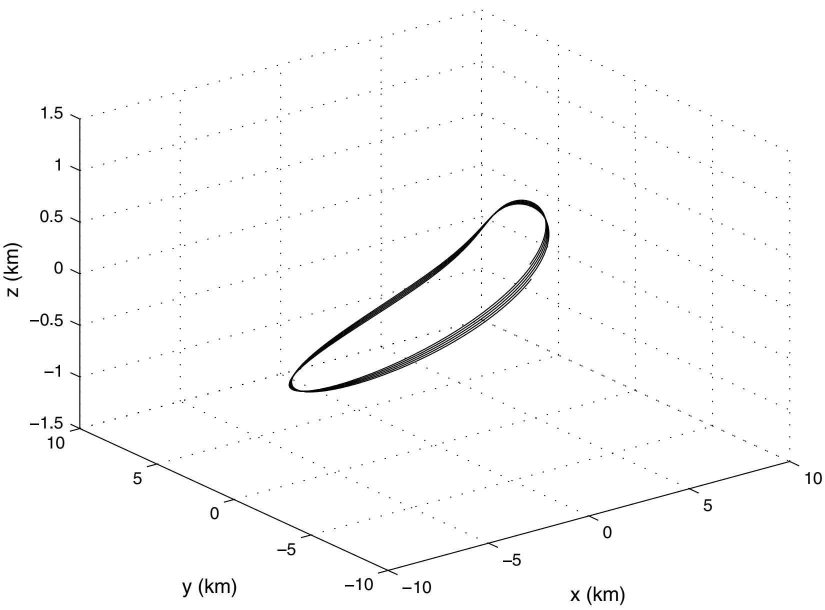


*Figure 7*. Nonlinear simulation of bounded relative motion trajectory in a sun-synchronous reference orbit.

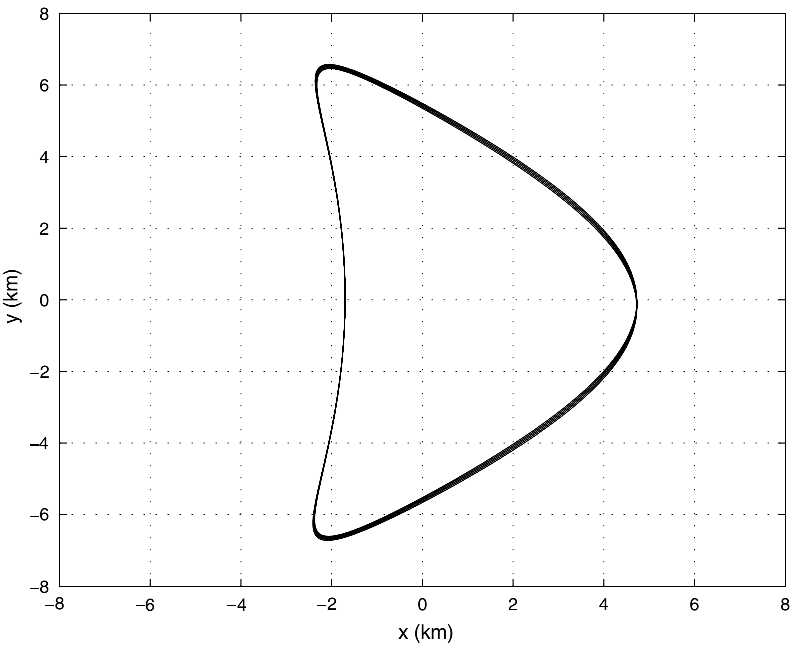


*Figure 8*. *x*-*y* projection of relative motion orbit with $J_2$ perturbations in sun-synchronous reference orbit.

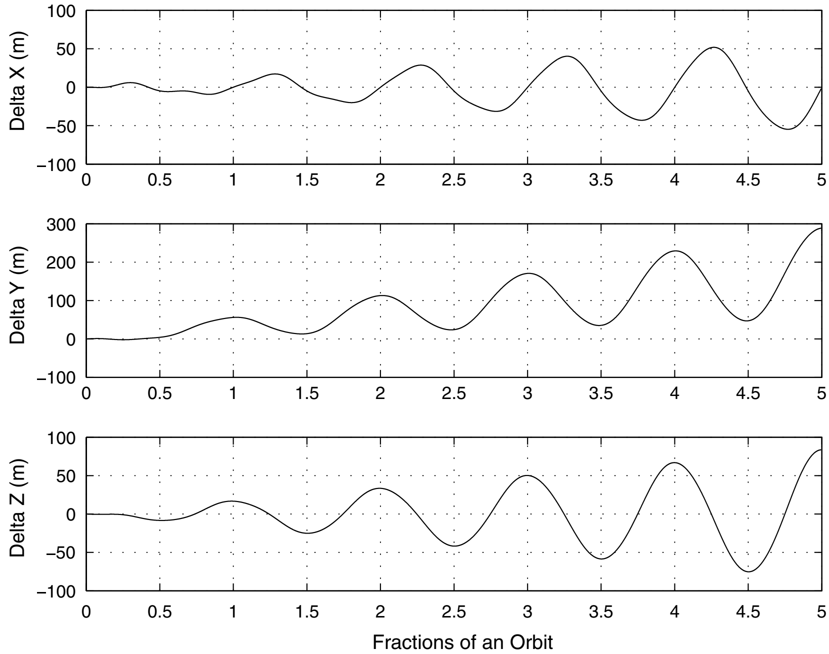


*Figure 9*. A comparison of the *x*–*y* projection of the relative motion orbit using the linearized, canonical equations and a full, inertial, nonlinear simulation over 5 orbits for a sun-synchronous reference orbit.

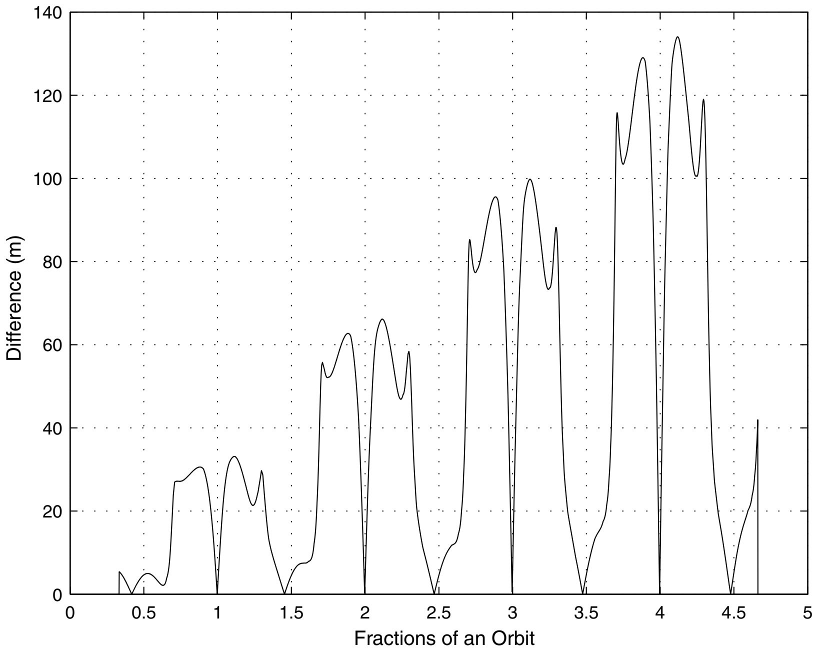


*Figure 10*. A comparison of the relative displacement between the linearized trajectory and that from a full, inertial, nonlinear simulation over 5 orbits for a sun-synchronous reference orbit.

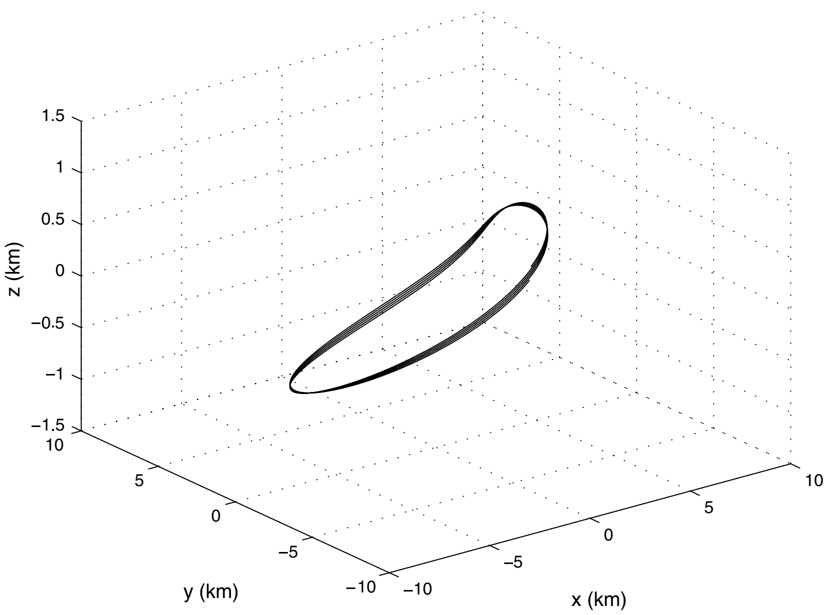


*Figure 11*. Nonlinear simulation of bounded, minimal drift relative motion trajectory in a sun-synchronous reference orbit.

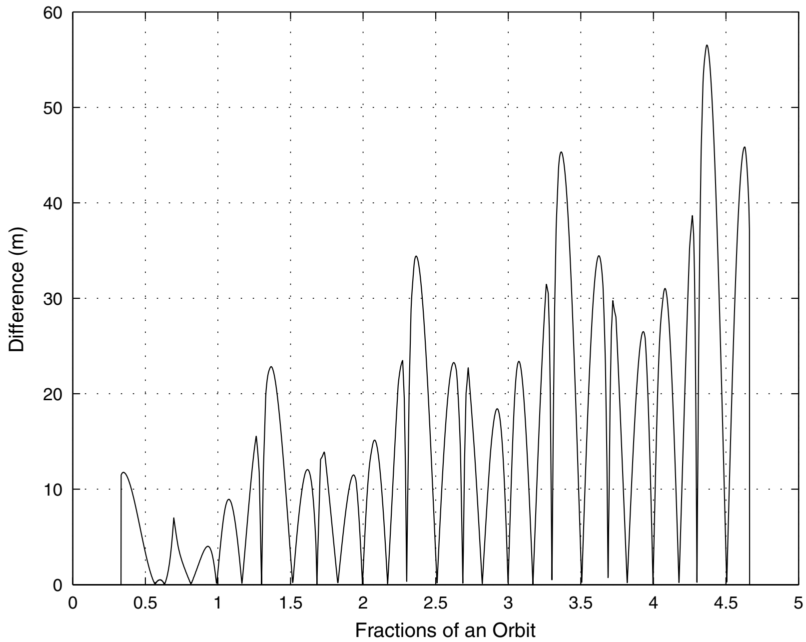


*Figure 12.* A comparison of the relative displacement between the linearized trajectory and that for a minimal drift relative motion over 5 orbits for a sun-synchronous reference orbit.

difference of the $x$–$y$ projection is displayed in Figure 6. The small, average $O(J_2^2)$ induced in-track drift is now on the order of only 4–5 m per orbit.

Figures 7 and 8 show the nonlinear simulation for a relative motion trajectory with the no-drift condition about a sun-synchronous reference orbit. This orbit could be used for a sun-synchronous satellite formation. Figure 9 shows the directly differenced error between this and the linear, canonical trajectory and Figure 10 shows the geometric difference. Unlike the low inclination orbit, here we see a rather large average drift due to the $O(J_2^2)$ terms of more than 20 m/orbit. Remarkably, that drift can be fairly easily reduced through a simple iteration of the nonlinear initial conditions. By using the no-drift condition as a starting point, we find a slight modification of the relative motion initial in-track velocity that significantly reduces the in-track drift in only 5 iterations. By changing the initial in-track velocity by just less than 0.006 m/s we can reduce the orbit drift to less than 5 m/orbit. Such an orbit is shown in Figure 11. The geometric difference between this nonlinear simulation and the original canonical, linearized orbit is shown in Figure 12.[3]

[3]For some high inclination orbits the $O(J_2^2)$ drift can become as high as 50–60 m per orbit. This can be reduced to the same low level via iteration.

## 6. Conclusions

This paper presented a new Hamiltonian framework for the analysis of spacecraft motion relative to a circular orbit in terms of canonical relative motion elements we termed ‘epicyclic’ elements. These epicyclic elements (and the equivalent ‘contact epicyclic’ elements) are constants of the linearized motion describing the relative satellite trajectory, similar to the orbital elements that describe motion in the two-body problem. These new elements are easily expressed in terms of the cartesian or polar initial conditions of the satellite motion. The value of this Hamiltonian approach is in the straightforward variation of parameters equations describing the change of the elements over time in the presence of perturbations or control. All equations are expressed entirely in the relative motion frame of reference, where most measurements are taken and where trajectory specification is most natural. While there are many applications of this approach, in this paper we calculated the effect of just one example perturbation–the $J_2$ Earth zonal harmonic. We were able to find very simple expressions for the relative motion and straightforward conditions for $J_2$-invariant orbits.

## Appendix A.

The Hamilton–Jacobi equation is a methodology for solving integrable dynamics problems via canonical transformations. We briefly describe the derivation here utilizing Goldstein (1980). Given a set of generalized coordinates $(q, \dot{q})$ and a Lagrangian $\mathcal{L}$, the canonical momenta are found via

$$p_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

The Hamiltonian is then given by the Legendre transformation:

$$\mathcal{H}(q, p, t) = \dot{q}_i p_i - \mathcal{L}(q, \dot{q}, t)$$

The $n$ second order equations of motion for the problem can then be alternatively written as $2n$ first order equations for $\dot{q}$ and $\dot{p}$ (Hamilton’s equations):

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

These canonical coordinates are not unique. If we consider a transformation of the phase space to a new set of coordinates $Q_i(q, p, t)$ and $P_i(q, p, t)$, we can ask for the class of transformations for which the new coordinates also satisfy Hamilton's equations on the new Hamiltonian $\mathcal{K}(Q, P, t) = \mathcal{H}(q(Q, P, \mathcal{H}), p(Q, P), t)$. Such a transformation is called canonical. A common approach to the transformation is via generating functions. For some function $F_2(q, P, t)$, a transformation is canonical provided that

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}$$

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_2}{\partial t}$$

The Hamilton–Jacobi problem comes from asking for the special canonical transformations for which $\mathcal{K} \equiv 0$ where the new canonical coordinates are constants of the motion. If such a transformation can be found, the equations of motion have been solved. The generating function for such a transformation is called Hamilton's Principle Function, $S(q, \alpha, t)$, where $\alpha_i$ is the new constant canonical momentum, $P_i$. This generating function can be found by setting the expression for $\mathcal{K}$ equal to zero while substituting from the transformation equations:

$$\mathcal{H}\left(q_1, \ldots, q_n; \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}; t\right) + \frac{\partial S}{\partial t} = 0$$

This is known as the Hamilton–Jacobi equation for $S$. Note that in the special case where $\mathcal{H}$ is independent of time it is a constant of the motion and can be set equal to $\alpha_1$. In this case, the HJ equation separates and we write Hamilton's principle function in terms of $W(q_i, \alpha_i)$, called Hamilton's characteristic function, and time

$$S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t) = W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n) - \alpha_1 t$$

The Hamilton–Jacobi equation for $W$ then reduces to:

$$\mathcal{H}\left(q_1, \ldots, q_n; \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = \alpha_1$$

## References

Alfriend, K. and Schaub, H.: 2000, 'Dynamics and control of spacecraft formations: challenges and some solutions', *J. Astronaut. Sci.* **48**(2), 249–267.

Alfriend, K. and Yan, H.: 2002, 'An orbital elements approach to the nonlinear formation flying problem', In: *International Formation Flying Symposium*, Toulouse, France.

Battin, R.: 1999, *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA.

Born, G. H.: 2001, 'Motion of a Satellite under the influence of an Oblate Earth'. Technical Report ASEN 3200, University of Colorado, Boulder.

Carter, T. and Humi, M.: 1987, 'Fuel-optimal rendezvous near a point in general Keplerian orbit', *J. Guidance, Control Dynam.* **10**(6), 567–573.

Clohessy, W. and Wiltshire, R.: 1960, 'Terminal guidance system for satellite rendezvous'. *J. Astronaut. Sci.* **27**(9), 653–678.

Efroimsky, M. and Goldreich, P.: 2003, 'Gauge symmetry of the N-body problem in the Hamilton–Jacobi approach', *J. Math. Phys.* **44**, 5958–5977.

Efroimsky, M. and Goldreich, P.: 2004, 'Gauge freedom in the N-body problem of celestial mechanics', *Astron. Astrophys.* **415**, 1187–1899.

Gim, D. and Alfriend, K.: 2001, 'The state transition matrix of relative motion for the perturbed non-circular reference Orbit', In: *Proceedings of the AAS/AIAA Space Flight Mechanics Meeting*, Santa Barbara, CA.

Goldstein, H.: 1980, *Classical Mechanics*. Addison-Wesley.

Gurfil, P. and Kasdin, N.: 2004, 'Nonlinear modeling and control of spacecraft relative motion in the configuration space', *J. Guidance, Control, Dynam.* **27**(1), 154–157.

Hill, G.: 1878, 'Researches in the lunar theory'. *Am. J Math.* **1**, 5–26.

Inalhan, G., Tillerson, M. and How, J.: 2002, 'Relative dynamics and control of spacecraft formations in eccentric orbits', *J. Guidance, Control Dynam.* **25**(1), 48–60.

Karlgaard, C. and Lutze, F.: 2001, 'Second-order relative motion equations'. In: *Proceedings of the AAS/AIAA Astrodynamics Conference*, Quebec City, Quebec.

Kasdin, N. and Gurfil, P.: 2004, 'Hamiltonian modeling of relative motion'. In: E. Belbruno, D. Folta and P. Gurfil (eds.), *Astrodynamics, Space Missions, and Chaos*, *The Annals of the New York Academy of Sciences*, Vol. 1017.

Namouni, F.: 1999, 'Secular interactions of coorbiting objects', *Icarus* **137**, 293–314.

Schaub, H. and Alfriend K.: 2001, '$J_2$ invariant relative orbits for spacecraft formations', *Celes. Mech. Dynam. Astron.* **79**, 77–95.

Schaub, H., Vadali, S. and Alfriend K.: 2000, 'Formation flying control using mean orbital elements', *J. Astronaut. Sci.* **48**(1), 69–87.

Scheeres, D., Hsiao, F. and Vinh, N.: 2003, 'Stabilizing motion relative to an unstable orbit: Applications to spacecraft formation flight', *J. Guidance, Control Dynam.* **26**(1), 62–73.

Vinti, J. P.: 1998, *Orbital and Celestial Mechanics*, *Progress in Astronautics and Aeronautics*, Vol. 177, AIAA.