

Relative Motion Between Elliptic Orbits: Generalized Boundedness Conditions and Optimal Formationkeeping

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Based on the concept of orbital commensurability, necessary and sufficient conditions are presented for bounded relative motion between any two spacecraft flying on elliptic Keplerian orbits. The proposed approach does not involve any simplifying assumptions regarding the relative dynamics but rather treats the general, nonlinear, eccentric relative motion problem. The methodology presented alleviates the difficulty in computing corrections to the linear equations of motion to account for nonlinearities and eccentricities. Instead of dealing with the local relative motion problem, the global relative motion problem is addressed by transforming the orbital resonance requirement into an energy-matching condition. The newly developed setup is then utilized to derive an optimal single-impulse formationkeeping maneuver based on relative state variables. The orbital elements interpretation of the optimal formationkeeping maneuver is also discussed.

Nomenclature

a	=	semimajor axis
\mathcal{E}	=	total specific mechanical energy
e	=	eccentricity
f	=	true anomaly
h	=	orbital angular momentum
\mathcal{I}	=	inertial frame
\mathcal{L}	=	leader frame
\mathbf{r}	=	position vector
\mathbf{v}	=	velocity vector
$\Delta \mathbf{v}$	=	velocity correction
θ	=	argument of latitude
λ	=	Lagrange multiplier
μ	=	gravitational constant
$\boldsymbol{\rho}$	=	relative position
Φ	=	cost functional
$\boldsymbol{\omega}$	=	angular velocity vector
$\ \cdot \ $	=	Euclidean vector norm

Subscripts and Superscripts

f	=	follower
i	=	value at time t_i
l	=	leader
0	=	initial condition
$-$	=	value before impulse
$+$	=	value after impulse

Introduction

THE problem of relative spacecraft dynamics dates back to the early 1960s, when Clohessy and Wiltshire (CW) published their celebrated work on satellite rendezvous.¹ The CW approximation assumed a circular reference orbit for the derivation of linear relative motion equations whose solution exhibited an in-track secular term. This drift can be avoided by a proper initialization of the CW equations to yield bounded relative motion. The CW equations constitute, however, only a local, autonomous approximation of the relative motion dynamics, and thus cannot be generalized

to describe the global relative motion between spacecraft flying on arbitrary Keplerian elliptic orbits.

Recognizing these limitations, many researchers have attempted to generalize the CW approximation. Higher-order extensions, which used orbital elements,² orbital element differences,³ canonical variables,⁴ primer vector theory,^{5,†} and curvilinear coordinates,⁶ were proposed. Generalizations of the CW equations that include reference orbit eccentricity have also been proposed.^{7,8} These works investigated the linearized problem of elliptic relative motion in order to calculate closed-form transition matrices for the relative velocity and position components. Recently, Vaddi et al.⁹ attempted to generalize the CW equations to include both nonlinearities and eccentricities by proposing a scheme for an incremental correction of the CW initial conditions.

This paper suggests a simple alternative for generating bounded relative motion between any two spacecraft flying on arbitrary elliptic Keplerian orbits. The underlying methodology relies on the concept of orbital commensurability, which, in the Keplerian case, naturally transforms into energy matching. We circumvent the difficulty in computing corrections to the CW equations⁹ and treat instead the global problem of Keplerian relative motion, and not the local, linearized problem,⁸ by formulating the energy matching condition in the rotating reference-orbit fixed frame. This formulation of orbital commensurability provides a single, simple, algebraic constraint on initial conditions guaranteeing bounded relative motion.

The new approach for generating bounded relative motion presented in this paper is a convenient stepping stone for developing an optimal formationkeeping scheme. In real-world scenarios, spacecraft formation initialization will almost always entail an initialization error.¹⁰ A formationkeeping maneuver must then be initiated in order to arrest the relative spacecraft drift. As relative spacecraft position control is usually performed utilizing relative measurements, it is of utmost importance to design formationkeeping maneuvers utilizing the relative state variables.

Previous works in the field have proposed either continuous thrusting for maintaining relative position^{11,12} or an impulsive correction of the mean orbital elements.¹³ The first approach is suitable for future missions equipped with low-thrust electric propulsion systems, whereas the latter approach does not utilize relative information.

In this paper, we develop a single-impulse formationkeeping maneuver based on energy matching. We use the inherent freedom of the energy-matching constraint in order to optimize the maneuver to

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†Data available online at <http://www.aisolutions.com/Resources/Papers/FormationFlyingPrimerVector.pdf>.

minimize fuel consumption. Although the treatment is carried out by using relative state variables only, we also provide an inertial insight into the resulting maneuver by performing a concomitant development of the optimal maneuver utilizing classical orbital elements.

This paper therefore derives a simple and unified framework for both initialization and initialization error correction that establishes bounded relative motion between any two spacecraft flying on arbitrary elliptic orbits without utilizing any simplifying assumptions regarding the relative dynamics.

Equations of Relative Motion

Consider two spacecraft orbiting a common gravitational body. One of the spacecraft, flying on a given reference orbit, is termed the leader and the other is referred to as the follower. The inertial, perturbation-free equations of motion of the leader, assuming Newtonian central-body gravitation, are given by

$$\ddot{\mathbf{r}}_l = -(\mu/r_l^3)\mathbf{r}_l \quad (1)$$

where $\mathbf{r}_l \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is the leader position vector in some inertial reference frame \mathcal{I} , and $r_l = \|\mathbf{r}_l\|$. In a similar fashion, the follower inertial equations of motion are

$$\ddot{\mathbf{r}}_f = -(\mu/r_f^3)\mathbf{r}_f \quad (2)$$

where $\mathbf{r}_f \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is the follower inertial position vector and $r_f = \|\mathbf{r}_f\|$. Let

$$\boldsymbol{\rho} = \mathbf{r}_f - \mathbf{r}_l \quad (3)$$

denote the position of the follower relative to the leader. Subtracting Eq. (1) from Eq. (2) yields the inertial relative acceleration

$$\ddot{\boldsymbol{\rho}} = -\mu/\|\mathbf{r}_l + \boldsymbol{\rho}\|^3 + (\mu/r_l^3)\mathbf{r}_l \quad (4)$$

It is useful to express the relative acceleration in a leader-fixed reference frame, as formation flying and other relative spacecraft operations utilize measurements of relative position and velocity. To that end, we shall adopt a Cartesian, rectangular, dextral rotating Euler–Hill coordinate system \mathcal{L} , centered at the leader. The fundamental plane of \mathcal{L} is the (instantaneous) orbital plane. The unit vector $\hat{\mathbf{x}}$ is directed from the leader spacecraft radially outward, $\hat{\mathbf{z}}$ is normal to the fundamental plane, and $\hat{\mathbf{y}}$ completes the setup.

To express the relative acceleration in frame \mathcal{L} , we recall that

$$\ddot{\boldsymbol{\rho}} = \frac{d^2[\boldsymbol{\rho}]_{\mathcal{L}}}{dt^2} + 2\mathcal{I}\boldsymbol{\omega}^{\mathcal{L}} \times \frac{d[\boldsymbol{\rho}]_{\mathcal{L}}}{dt} + \frac{d\mathcal{I}\boldsymbol{\omega}^{\mathcal{L}}}{dt} \times [\boldsymbol{\rho}]_{\mathcal{L}} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{L}}(\mathcal{I}\boldsymbol{\omega}^{\mathcal{L}} \times [\boldsymbol{\rho}]_{\mathcal{L}}) \quad (5)$$

where the operator $d(\cdot)$ denotes differentiation in frame \mathcal{L} and $\mathcal{I}\boldsymbol{\omega}^{\mathcal{L}}$ denotes the angular velocity vector of frame \mathcal{L} relative to frame \mathcal{I} . As $\mathcal{I}\boldsymbol{\omega}^{\mathcal{L}}$ is normal to the orbital plane, we can write

$$\mathcal{I}\boldsymbol{\omega}^{\mathcal{L}} = [0, 0, \dot{\theta}_l]^T \quad (6)$$

where θ_l is the leader's argument of latitude and $\dot{\theta}_l$ is the leader's angular velocity, given by

$$\dot{\theta}_l = \sqrt{\mu/a_l^3(1-e_l^2)^3(1+e_l \cos f_l)} \quad (7)$$

where a_l , e_l , and f_l are the leader's semimajor axis, eccentricity, and true anomaly. In the disturbance-free case discussed here,

$$\dot{\theta}_l = \dot{f}_l \quad (8)$$

The position vector of the leader spacecraft in \mathcal{L} is given by

$$[\mathbf{r}_l]_{\mathcal{L}} = [r_l, 0, 0]^T \quad (9)$$

where r_l can be expressed by the polar conic equation

$$r_l = \frac{a_l(1-e_l^2)}{1+e_l \cos f_l} \quad (10)$$

and the relative position vector is written as

$$[\boldsymbol{\rho}]_{\mathcal{L}} = [x, y, z]^T \quad (11)$$

Substituting Eqs. (4), (6), and (11) into Eq. (5) yields the following component-wise equations for relative motion:

$$\ddot{x} - 2\dot{\theta}_l \dot{y} - \ddot{\theta}_l y - \dot{\theta}_l^2 x = -\frac{\mu(r_l+x)}{[(r_l+x)^2+y^2+z^2]^{\frac{3}{2}}} + \frac{\mu}{r_l^2} \quad (12)$$

$$\ddot{y} + 2\dot{\theta}_l \dot{x} + \ddot{\theta}_l x - \dot{\theta}_l^2 y = -\frac{\mu y}{[(r_l+x)^2+y^2+z^2]^{\frac{3}{2}}} \quad (13)$$

$$\ddot{z} = -\frac{\mu z}{[(r_l+x)^2+y^2+z^2]^{\frac{3}{2}}} \quad (14)$$

Equations (12–14) with Eqs. (7) and (10) substituted for $\dot{\theta}_l$ and r_l , respectively, constitute a sixth-order system of nonlinear nonautonomous differential equations. These equations admit a single relative equilibrium at $x = y = z = 0$, meaning that the follower spacecraft will appear stationary in the leader frame if and only if their positions coincide on a given elliptic orbit. Thus, maintaining a constant relative position between spacecraft flying on elliptic orbits is impossible unless continuous thrusting is applied. Nevertheless, as we shall subsequently illustrate, impulsive formationkeeping can provide bounded relative motion. To this end, the next section develops a necessary and sufficient condition for bounded relative motion between any two spacecraft flying on arbitrary elliptic orbits.

Orbital Commensurability and the Energy-Matching Condition

To derive an impulsive formationkeeping scheme, it is important to inquire under what conditions Eqs. (12–14) provide bounded solutions. This query has a straightforward answer, derived from the Keplerian topology of the orbits: if both vehicles follow Keplerian elliptic orbits, then their separation cannot grow unboundedly. However, if the periods of the spacecraft orbits do not commensurate, periodicity or quasi-periodicity will be exhibited only on very large timescales, and hence, on short timescales the motion will appear unbounded (as indicated by the CW equations). It is obvious, on the other hand, that if the orbits of the spacecraft do commensurate then periodicity will be exhibited in the form of resonant relative motion. We shall study herein the 1:1 relative motion resonance, as it is the most practical and meaningful for spacecraft formation flying and rendezvous.

Because the periods of Keplerian elliptic orbits are uniquely determined by the orbital energy, we can transform the period commensurability requirement into an energy-matching condition. We say that two spacecraft on arbitrary elliptic orbit satisfy the energy-matching condition if their orbital energies are equal, or, equivalently, if the semimajor axes of their orbits are identical.

We shall formulate the energy-matching condition in the rotating frame \mathcal{L} as follows. The velocity of the follower in the rotating frame satisfies

$$\mathbf{v}_f = \frac{d}{dt}[\boldsymbol{\rho}]_{\mathcal{L}} + \frac{d}{dt}[\mathbf{r}_l]_{\mathcal{L}} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{L}} \times [\mathbf{r}_l]_{\mathcal{L}} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{L}} \times [\boldsymbol{\rho}]_{\mathcal{L}} \quad (15)$$

Substituting Eqs. (6), (9), and (11) into Eq. (15) yields

$$\mathbf{v}_f = \begin{bmatrix} \dot{x} - \dot{\theta}_l y + \dot{r}_l \\ \dot{y} + \dot{\theta}_l(x+r_l) \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \quad (16)$$

where, utilizing Eq. (8),

$$\begin{aligned} \dot{r}_l &= \frac{d}{dt} \left[\frac{a_l(1-e_l^2)}{1+e_l \cos f_l} \right] = \dot{f}_l \frac{d}{df_l} \left[\frac{a_l(1-e_l^2)}{1+e_l \cos f_l} \right] \\ &= \dot{\theta}_l \frac{a_l e_l (1-e_l^2) \sin f_l}{(1+e_l \cos f_l)^2} \end{aligned} \quad (17)$$

Substituting for $\dot{\theta}_l$ from Eq. (7) gives

$$\dot{r}_l = e_l \sin f_l \sqrt{\mu/a_l(1-e_l^2)} \quad (18)$$

The total specific energy of the follower spacecraft comprises the kinetic and potential energies,

$$\begin{aligned} \mathcal{E}_f &= \frac{1}{2} \|\mathbf{v}_f\|^2 - \mu/r_f = \frac{1}{2} v_f^2 - \mu/r_f = \frac{1}{2} (v_x^2 + v_y^2 + v_z^2) - \mu/r_f \\ &= \frac{1}{2} \{ (\dot{x} - \dot{\theta}_l y + \dot{r}_l)^2 + [\dot{y} + \dot{\theta}_l(x+r_l)]^2 + \dot{z}^2 \} \\ &\quad - \mu / \sqrt{(r_l+x)^2 + y^2 + z^2} \end{aligned} \quad (19)$$

The total energy of the leader is given by

$$\mathcal{E}_l = -\mu/2a_l \quad (20)$$

The energy-matching condition, guaranteeing a 1:1 resonant relative motion, can be therefore stated as follows:

$$\begin{aligned} \frac{1}{2} \{ (\dot{x} - \dot{\theta}_l y + \dot{r}_l)^2 + [\dot{y} + \dot{\theta}_l(x+r_l)]^2 + \dot{z}^2 \} \\ - \mu / \sqrt{(r_l+x)^2 + y^2 + z^2} = -\mu/2a_l \end{aligned} \quad (21)$$

Equation (21) must be satisfied for all times $t \geq t_0$, therefore constituting a generalization of existing results¹⁴ by expressing energy matching at any point along the follower's orbit. An important special case of Eq. (21), also discussed in previous works,¹⁴ is formation initialization, that is, for $t = t_0$, Eq. (21) indicates how to initialize relative spacecraft orbits to get bounded relative motion. To see this, denote

$$\begin{aligned} f_l(t_0) &= f_0, & x(f_0) &= x_0, & y(f_0) &= y_0, & z(f_0) &= z_0 \\ \dot{x}(f_0) &= \dot{x}_0, & \dot{y}(f_0) &= \dot{y}_0, & \dot{z}(f_0) &= \dot{z}_0 \\ \dot{\theta}_l(f_0) &= \dot{\theta}_0, & r_{l0} &= r_l(f_0), & \dot{r}_{l0} &= \dot{r}_l(f_0) \end{aligned} \quad (22)$$

Substituting Eq. (22) into Eq. (21) shows that to get a 1:1 bounded relative motion it is necessary and sufficient to initialize the formation according to the constraint

$$\begin{aligned} \frac{1}{2} \{ (\dot{x}_0 - \dot{\theta}_0 y_0 + \dot{r}_{l0})^2 + [\dot{y}_0 + \dot{\theta}_0(x_0+r_{l0})]^2 + \dot{z}_0^2 \} \\ - \mu / \sqrt{(r_{l0}+x_0)^2 + y_0^2 + z_0^2} = -\mu/2a_l \end{aligned} \quad (23)$$

Equation (23) constitutes the most general expression for the initial conditions guaranteeing bounded motion (in the 1:1 commensurability sense) between any two spacecraft on arbitrary Keplerian elliptic orbits. It is formulated in the relative, leader-centered reference frame and is therefore of utmost importance for the design of relative spacecraft orbits, generalizing the CW conditions for bounded motion as well as considerably extending the recently developed extension of the CW initialization.⁹ Equation (23) inherently accommodates orbital eccentricities and nonlinearities and hence circumvents the need to calculate complex corrections to the CW initialization as suggested by Vaddi et al.⁹

Equation (23) also shows that the CW initialization for locally bounded motion, $\dot{y}_0 = -2x_0$, is neither necessary nor sufficient for bounded motion in the global sense. In fact, the CW initialization constitutes merely a special case of Eq. (23). One can find initial conditions for which $\dot{y}_0 \neq -2x_0$, but energy matching will still be satisfied. This is illustrated by example 1.

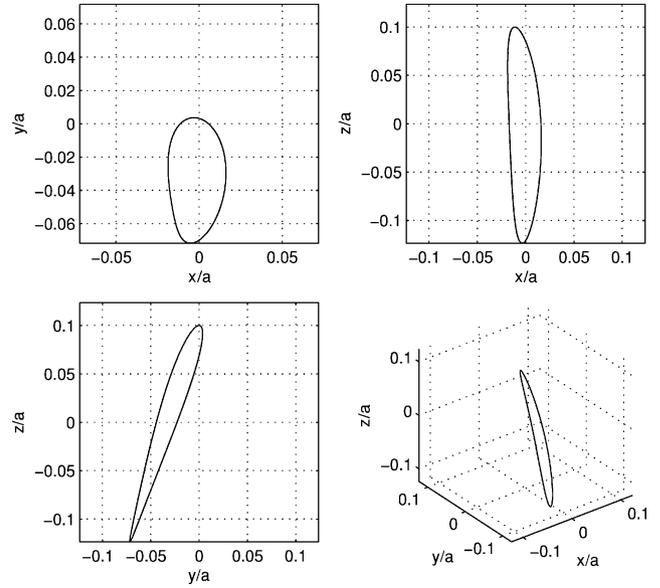


Fig. 1 Relative motion in the configuration space exhibits bounded orbits.

Example 1: Consider a leader spacecraft on an elliptic orbit. Normalize positions by a_l and rates by $\sqrt{(\mu/a_l^3)}$ so that $a_l = \mu = 1$. Let

$$\begin{aligned} y_0 &= 0, & z_0 &= 0.1, & \dot{x}_0 &= 0.02, & \dot{y}_0 &= 0.02 \\ \dot{z}_0 &= 0, & f_0 &= 0, & e_l &= 0.1 \end{aligned} \quad (24)$$

Find x_0 that guarantees a 1:1 bounded relative motion.

We first calculate r_{l0} , \dot{r}_{l0} , and $\dot{\theta}_0$:

$$r_{l0} = \frac{1-e_l^2}{1+e_l \cos f_0} = \frac{1-0.1^2}{1+0.1} = 0.9$$

$$\dot{r}_{l0} = e_l \sin f_0 \sqrt{\frac{1}{1-e_l^2}} = 0$$

$$\dot{\theta}_0 = \sqrt{\frac{1}{(1-e_l^2)^3}} (1+e_l \cos f_0)^2 = 1.22838$$

Upon substitution into Eq. (23), we obtain a sixth-order equation for x_0 :

$$\begin{aligned} 2.2768x_0^6 + 12.4431x_0^5 + 31.3762x_0^4 + 45.46062x_0^3 \\ + 39.5905x_0^2 + 19.5344x_0 + 0.2151 = 0 \end{aligned}$$

There are two real solutions:

$$x_{01} = -0.01127, \quad x_{02} = -1.8059$$

These initial conditions on the radial separation of the spacecraft will guarantee 1:1 bounded motion, although they violate the CW requirement ($\dot{y}_0 = -2x_0$). To illustrate the resulting orbits, we simulated the equations of motion using the initial conditions (24) and $x_0 = x_{01}$. The results are depicted by Fig. 1, which shows the three-dimensional orbit (bottom-right panel) and the xy , xz , and yz projections. The relative position is bounded, and periodicity is obtained by satisfying the 1:1 commensurability conditions of Eq. (23).

Impulsive Formationkeeping for Bounded Relative Motion

Equation (23) constitutes a necessary and sufficient condition for 1:1 resonant relative motion. However, in practice, because of initialization errors, this constraint cannot be satisfied exactly, causing

a “drift” in relative position. To compensate for such errors, the follower spacecraft must perform a drift-arresting formationkeeping maneuver. In most cases, the maneuver will be carried out using the onboard impulsive propulsion system, providing impulsive thrust pulses.

We shall develop herein a single-impulse, energy-matching-based formationkeeping maneuver strategy aimed at correcting the relative position drift while consuming minimum fuel. To this end, let the position initialization errors be denoted by $\delta\rho_0 = [\delta x_0, \delta y_0, \delta z_0]^T$ and the velocity initialization errors be $\delta v_0 = [\delta \dot{x}_0, \delta \dot{y}_0, \delta \dot{z}_0]^T$. The actual initial conditions are thus

$$\rho_0^\delta = \rho_0 + \delta\rho_0, \quad v_0^\delta = v_0 + \delta v_0 \quad (25)$$

The impulsive maneuver will be represented by a relative velocity correction performed by the follower spacecraft in the rotating Euler–Hill frame, $[\Delta v]_{\mathcal{L}} = [\Delta v_x, \Delta v_y, \Delta v_z]^T$. We shall subsequently omit the subscript \mathcal{L} to facilitate notation. Assuming that the impulsive maneuver is initiated at time t_i , the required velocity correction components can be determined by requiring that the follower spacecraft’s total energy after the correction \mathcal{E}_f^+ matches the total energy of the leader:

$$\mathcal{E}_f^+ = \frac{1}{2} \{ (v_x^- + \Delta v_x)^2 + (v_y^- + \Delta v_y)^2 + [v_z^- + \Delta v_z]^2 \} - \mu/r_f^- = -\mu/2a_l \quad (26)$$

where

$$v_x^- = \dot{x}_i^- - \dot{\theta}_i^- y_i^- + \dot{r}_i^- \quad (27)$$

$$v_y^- = \dot{y}_i^- + \dot{\theta}_i^- (x_i^- + r_i^-) \quad (28)$$

$$v_z^- = \dot{z}_i^- \quad (29)$$

$$r_f^- = \sqrt{[(r_l)_i^- + x_i^-]^2 + (y_i^-)^2 + (z_i^-)^2} \quad (30)$$

and $(\cdot)_i^-$ indicates values prior to the impulsive maneuver at $t = t_i$. Apparently, we have three degrees of freedom for choosing the velocity correction components but only one constraint, Eq. (26). This means that we are free to choose two degrees of freedom. As we are interested in saving fuel, the extra freedom will be utilized in order to minimize the required fuel consumption via solution of an optimization problem. We note that if t_i is given, the underlying optimization problem is static, as it involves parameters defined at a single time instant t_i . Otherwise, a dynamic optimization problem must be solved or approximated by successive static optimization procedures for each t_i . We shall assume hereafter that the t_i is predetermined based on operational consideration and subsequently solve a static optimization problem. We shall further comment on selecting t_i in the next section.

Static parameter optimization problems with equality constraints can be straightforwardly solved utilizing Lagrange multipliers. In our case, the optimization problem can be stated as follows:

Find an optimal impulsive maneuver Δv^* , satisfying

$$\begin{aligned} \Delta v^* &= \min_{\Delta v} \|\Delta v\|^2 \\ \text{s.t.} \\ \mathcal{E}_f^+ &= -\mu/2a_l \end{aligned} \quad (31)$$

Augmenting the cost functional with the equality constraint using the Lagrange multiplier λ yields

$$\Phi = \|\Delta v\|^2 + \lambda(\mathcal{E}_f^+ + \mu/2a_l) \quad (32)$$

The necessary and sufficient conditions for the existence of minima are

$$\frac{\partial \Phi}{\partial (\Delta v)} = 0 \quad (33)$$

$$\frac{\partial^2 \Phi}{\partial (\Delta v)^2} > 0 \quad (34)$$

Equations (33) and (26) constitute a system of four quadratic algebraic equations for the four variables Δv_x , Δv_y , Δv_z , and λ :

$$2\Delta v_x + \lambda(v_x^- + \Delta v_x) = 0 \quad (35)$$

$$2\Delta v_y + \lambda(v_y^- + \Delta v_y) = 0 \quad (36)$$

$$2\Delta v_z + \lambda(v_z^- + \Delta v_z) = 0 \quad (37)$$

$$\frac{1}{2} [(v_x^- + \Delta v_x)^2 + (v_y^- + \Delta v_y)^2 + (v_z^- + \Delta v_z)^2] - \mu/r_f^- + \mu/2a_l = 0 \quad (38)$$

solution whereof yields

$$\frac{\Delta v_x}{v_x^-} = \frac{\Delta v_y}{v_y^-} = \frac{\Delta v_z}{v_z^-} = -1 \pm \frac{1}{v_f^-} \sqrt{\frac{\mu(2a_l - r_f^-)}{a_l r_f^-}} \quad (39)$$

$$\lambda = -2 \pm 2v_f^- \sqrt{\frac{a_l r_f^-}{\mu(2a_l - r_f^-)}} \quad (40)$$

Real and finite solutions are obtained provided that $r_f^- \leq 2a_l$.

The Hessian is

$$\frac{\partial^2 \Phi}{\partial (\Delta v)^2} = \begin{bmatrix} \lambda + 2 & 0 & 0 \\ 0 & \lambda + 2 & 0 \\ 0 & 0 & \lambda + 2 \end{bmatrix} \quad (41)$$

which is positive definite if and only if $\lambda > -2$. Because $v_f^- > 0$, $r_f^- > 0$, $\mu > 0$, and $a_l > 0$, we conclude that only the first of solutions (40) corresponds to a minimum

$$\lambda = -2 + 2v_f^- \sqrt{\frac{a_l r_f^-}{\mu(2a_l - r_f^-)}} \quad (42)$$

The corresponding optimal velocity corrections are therefore

$$\frac{\Delta v_x^*}{v_x^-} = \frac{\Delta v_y^*}{v_y^-} = \frac{\Delta v_z^*}{v_z^-} = -1 + \frac{1}{v_f^-} \sqrt{\frac{\mu(2a_l - r_f^-)}{a_l r_f^-}} \quad (43)$$

and the minimum total velocity correction is

$$\Delta v^* = \sqrt{(\Delta v_x^*)^2 + (\Delta v_y^*)^2 + (\Delta v_z^*)^2} = v_f^- - \sqrt{\frac{\mu(2a_l - r_f^-)}{r_f^- a_l}} \quad (44)$$

Example 2: Consider a leader spacecraft on an elliptic orbit. Normalize positions by a_l and rates by $\sqrt{\mu/a_l^3}$ so that $a_l = \mu = 1$. Let the nominal initial conditions be as in Eq. (24) with $x_0 = -0.01127$. Assume that the initialization errors are

$$\begin{aligned} \delta x_0 &= 0.001, & \delta y_0 &= 0.001, & \delta z_0 &= 0.01 \\ \delta \dot{x}_0 &= 0, & \delta \dot{y}_0 &= 0, & \delta \dot{z}_0 &= 0 \end{aligned} \quad (45)$$

Compute the minimum-fuel maneuver required to obtain a 1:1 bounded relative motion assuming that the maneuver is to be applied after one orbital period of the leader spacecraft.

After a single orbital period, $t_i = 1$ (in normalized units). Utilizing the initial conditions (24) and the equations of motion (12–14), we have

$$\begin{aligned} x_i^- &= -0.015374, & y_i^- &= -0.084596, & z_i^- &= 0.109547 \\ \dot{x}_i^- &= 0.00994, & \dot{y}_i^- &= 0.021792, & \dot{z}_i^- &= 0.011765 \\ \dot{\theta}_i^- &= 1.22838, & r_i^- &= 0.9, & \dot{r}_i^- &= 0 \end{aligned} \quad (46)$$

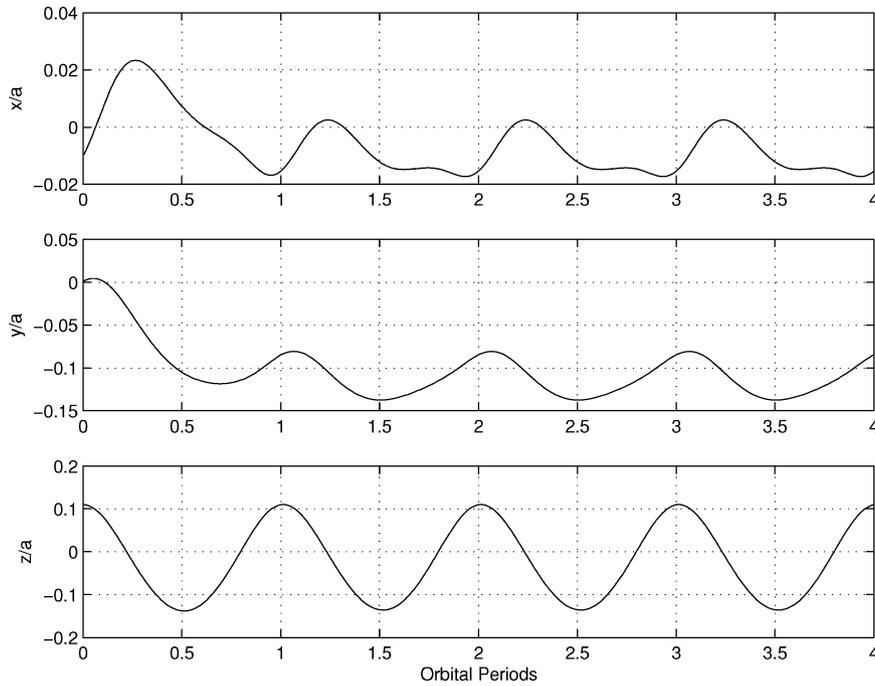


Fig. 2 Time histories of the normalized relative position components show periodic behavior following the formationkeeping maneuver initiated after one orbital period.

Substituting into Eqs. (27–30) yields

$$\begin{aligned} v_x^- &= 0.11386, & v_y^- &= 1.10845 \\ v_z^- &= 0.01177, & r_f^- &= 0.89538 \end{aligned} \quad (47)$$

Substituting again into Eq. (43) yields the optimal formationkeeping maneuver

$$\begin{aligned} \Delta v_x^* &= -0.00037144, & \Delta v_y^* &= -0.00361606 \\ \Delta v_z^* &= -0.00003838 \end{aligned} \quad (48)$$

which results in the total normalized Δv

$$\Delta v^* = \sqrt{(\Delta v_x^*)^2 + (\Delta v_y^*)^2 + (\Delta v_z^*)^2} = 0.0036353 \quad (49)$$

Figures 2–4 depict the results of a simulation performed utilizing the preceding values. Figure 2 shows the time histories of the normalized relative position components. Time is normalized by the leader's orbital period, so that the normalized time is $T = t/[2\pi\sqrt{(\mu/a_f^3)}]$. The impulsive formationkeeping maneuver is carried out at $T = 1$. Following this maneuver, the position components converge to a periodic motion reflecting the 1:1 relative motion resonance. Note the transient response leading to periodicity.

Figure 3 shows the three-dimensional relative orbit in the Euler–Hill frame utilizing normalized position components (bottom-right panel) and the projections of the orbit on the xy , xz , and yz planes. The initial relative drift is arrested at $T = 1$, and a bounded relative motion results.

Figure 4 exhibits a magnification of the time history of the normalized relative velocity components \dot{x} , \dot{y} , \dot{z} at the vicinity of the impulsive maneuver and the total energy of the follower spacecraft (bottom-right panel). The discontinuity in the relative velocity component is a result of the impulsive velocity change. Note that the initial energy of the follower in normalized units is $\mathcal{E}_f = -0.496$, which differs from the total energy of the leader, $\mathcal{E}_l = -0.5$. The impulsive maneuver decreases the total energy of the follower by 0.004, matching it to the leader's energy and establishing a 1:1 bounded motion.

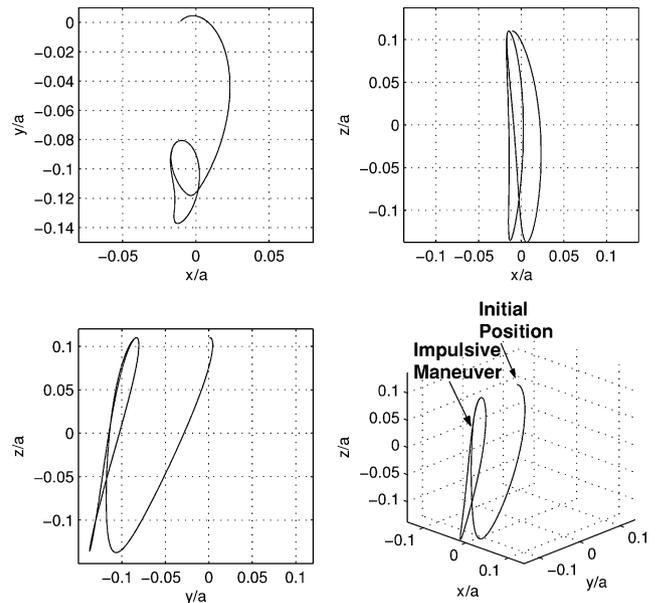


Fig. 3 Relative motion in the configuration space exhibits bounded orbits following the formationkeeping maneuver.

Orbital-Elements Outlook on Optimal Formationkeeping

In the preceding section, we developed an optimal formationkeeping maneuver for the follower spacecraft using an impulsive velocity correction $[\Delta v]_{\mathcal{L}}$ formulated in a leader-fixed Euler–Hill frame. This maneuver has a straightforward interpretation if formulated in a follower-fixed frame. To see this, we utilize Gauss's variational equations.¹⁵ Letting $[u_r, u_\theta, u_h]^T$ be the follower's thrust acceleration components in the radial, in-track, and normal directions, respectively, the variational equation for the semimajor axis of the follower spacecraft a_f is¹⁵

$$\dot{a}_f = (2a_f^2/h_f)[e_f \sin f_f u_r + (1 + e_f \cos f_f)u_\theta] \quad (50)$$

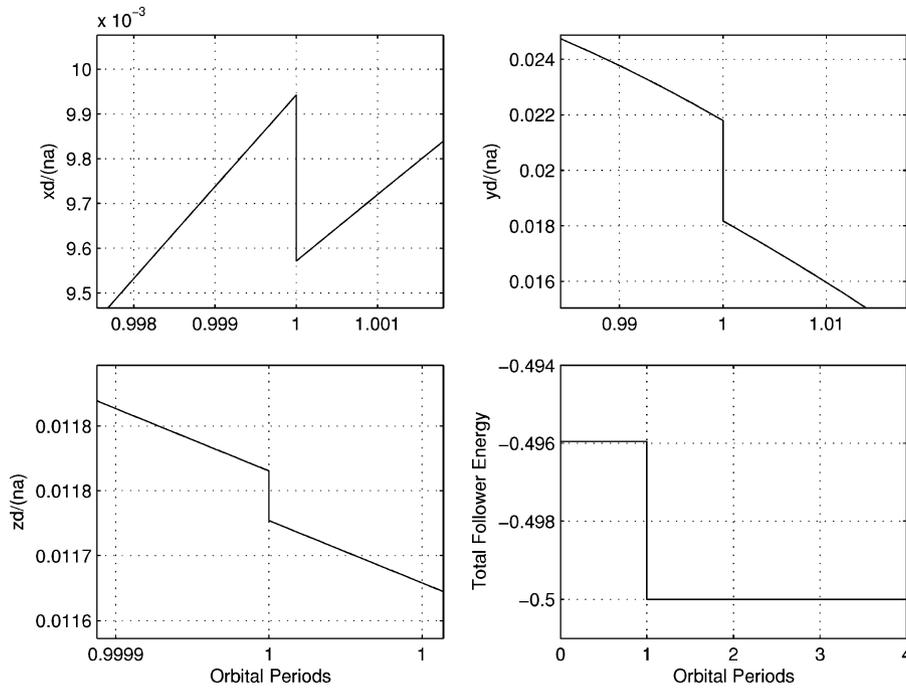


Fig. 4 Magnification of the relative velocity components at the vicinity of the formationkeeping maneuver shows the applied velocity impulse, decreasing the total energy of the follower to match the total energy of the leader (bottom-right graph).

where f_f , e_f , and $h_f = \sqrt{[\mu a_f(1 - e_f^2)]}$ are the follower's true anomaly, eccentricity, and orbital angular momentum, respectively. An impulsive maneuver aimed at matching the semimajor axis of the follower to the leader's orbit semimajor axis can be therefore expressed as

$$\Delta a_f = (2a_f^2/h_f)[e_f \sin f_f \Delta v_r + (1 + e_f \cos f_f) \Delta v_\theta] \quad (51)$$

where $[\Delta \mathbf{v}]_{\mathcal{F}} = \Delta \mathbf{v}_f = [\Delta v_r, \Delta v_\theta, \Delta v_h]^T$ is the impulsive velocity correction vector in a follower-fixed rotating frame. Let us find a minimum-energy impulsive maneuver by using an optimization problem formulated similarly to Eq. (31):

$$\Delta v_f^* = \min_{\Delta v_r, \Delta v_\theta} \Delta v_r^2 + \Delta v_\theta^2 \quad (52)$$

s.t.

$$\Delta a_f = (2a_f^2/h_f)[e_f \sin f_f \Delta v_r + (1 + e_f \cos f_f) \Delta v_\theta] \quad (53)$$

Using a Lagrange multiplier λ , the augmented cost is

$$\Phi = \Delta v_r^2 + \Delta v_\theta^2 + \lambda \left\{ (2a_f^2/h_f)[e_f \sin f_f \Delta v_r + (1 + e_f \cos f_f) \Delta v_\theta] \right\} \quad (54)$$

Note that Φ is time dependent via the follower's true anomaly f_f . The necessary and sufficient conditions for extrema are therefore

$$\frac{\partial \Phi}{\partial (\Delta \mathbf{v}_f)} = 0 \quad (55)$$

$$\frac{\partial \Phi}{\partial f_f} = 0 \quad (56)$$

$$\frac{\partial^2 \Phi}{\partial [\Delta \mathbf{v}_f, f_f]^2} > 0 \quad (57)$$

The necessary conditions, Eqs. (55) and (56), yield

$$\Delta v_r + \lambda a_f^2 e_f \sin f_f / h_f = 0 \quad (58)$$

$$\Delta v_\theta + \lambda a_f^2 (1 + e_f \cos f_f) / h_f = 0 \quad (59)$$

$$\lambda a_f^2 (e_f \cos f_f \Delta v_r - e_f \sin f_f \Delta v_\theta) = 0 \quad (60)$$

Equations (58–60) together with constraint (53) constitute a system of four algebraic equations for the unknowns Δv_r , Δv_θ , λ , f_f . There are two possible solutions:

$$\begin{aligned} \Delta v_r &= 0, & \Delta v_\theta &= \frac{h_f \Delta a_f}{2a_f^2(1 + e_f)} \\ \lambda &= -\frac{h_f^2 \Delta a_f}{2a_f^4(1 + e_f)^2}, & f_f &= 0 \end{aligned} \quad (61)$$

$$\begin{aligned} \Delta v_r &= 0, & \Delta v_\theta &= \frac{h_f \Delta a_f}{2a_f^2(1 - e_f)} \\ \lambda &= -\frac{h_f^2 \Delta a_f}{2a_f^4(1 - e_f)^2}, & f_f &= \pi \end{aligned} \quad (62)$$

Evaluating the Hessian shows that Eq. (61) is the minimum solution, thus reassuring the well-known fact that an optimal semimajor axis correction maneuver should be performed at periapsis ($f_f = 0$):

$$\Delta \mathbf{v}_f^* = [0, \Delta v_\theta(f_f = 0), 0]^T \quad (63)$$

Hence, we have obtained an indirect indication of the optimal maneuver initiation time t_i mentioned in the preceding section: unless operational constraints require otherwise, the drift-arresting maneuver should be performed at the perigee of the follower spacecraft.

We shall now show that the magnitude of the velocity correction in the follower's frame, Eq. (63), equals the magnitude of the velocity correction in the leader's frame, Eq. (44). First, we write the vis vis equation to obtain an expression for the leader's velocity v_l at a given position r_l :

$$v_l = \sqrt{\frac{2(a_l - r_l)}{a_l r_l}} \quad (64)$$

We next note that the orbits of the leader and the follower must intersect at the point where the impulsive maneuver is applied, so that

$$v_l = \sqrt{\frac{2(a_l - r_f^-)}{a_l r_f^-}} \quad (65)$$

Equation (44) can be therefore rewritten as

$$\Delta v^* = v_f^- - v_l \quad (66)$$

Utilizing the fact that at perigee

$$\Delta v_f^* = \Delta v_\theta = v_f^- - v_l \quad (67)$$

we can conclude that, as expected,

$$\Delta v^* = \Delta v_f^* \quad (68)$$

Conclusions

The main conclusion from this paper is that bounded relative motion between any two spacecraft flying on elliptic Keplerian orbits can be straightforwardly found from the energy matching condition and that there is no need to correct the linear Clohessy and Wiltshire initialization to account for nonlinearities and eccentricities. By dealing with the full nonlinear problem, global conditions for bounded motion can be found which naturally accommodate the nonlinearity of the Keplerian relative motion equations as well as the reference orbit eccentricity.

We also conclude that orbital commensurability, guaranteeing bounded relative motion, can be reestablished in spite of initialization errors by applying a single thrust impulse, which can be calculated and optimized using relative state variables and has a simple interpretation in terms of the follower's orbital elements. This single-impulse maneuver also has considerable operational advantages over multiple impulses.

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