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Euler Parameters as Nonsingular Orbital Elements in Near-Equatorial Orbits

Pini Gurfil*
Technion—Israel Institute of Technology, 32000 Haifa, Israel

Introduction

THE analysis of Keplerian motion is most often formulated using classical orbital elements. When perturbations are introduced, time variation of these elements can be calculated based on a standard variation-of-parameters (VOP) procedure.

Under the classical paradigm, Euler angles are used to parameterize the orbital plane orientation. However, due to the inherent singularity of the Euler angles, the variational equations become singular for zero eccentricities and/or zero inclinations. These singularities require a regularization procedure, which results in a modified set of orbital elements.

The most commonly used set of nonsingular elements is the equinoctial elements,[1–4] which are ubiquitous in engineering applications.[5] Astronomers and physicists, however, prefer the Kustaanheimo–Stiefel (KS)[6,7] orbital elements. Interestingly, it has been shown that the KS elements form a quaternion,[8] comprising complex functions of both orbital elements and the inertial position and velocity.

Is there a more physically meaningful methodology to regularize the perturbed equations of motion? In this Note, we present a regularization approach that is more natural than existing approaches, providing better insight into the modeling of perturbed Keplerian motion. Because the singularity of the variational equations stems from the use of Euler angles, we suggest replacing the Euler angles with Euler parameters. Although this technique is very common in rigid-body attitude dynamics and control,[9] it has been less explored in orbital mechanics. Similar to the KS elements, Euler parameters form a quaternion. However, the KS elements have little physical meaning, whereas Euler parameters are well-defined geometric quantities. Also, it is practically impossible to obtain planetary equations using the KS paradigm via a VOP procedure.

In this work, we develop the planetary equations in terms of Euler parameters, replacing the inclination, argument of periapsis, and right ascension of the ascending node, without resorting to hyperimaginary (quaternion) algebra. The resulting variational equations are expressed in Lagrange form. We illustrate the use of the new elements by investigating the effect of zonal gravitational harmonics, rewritten in terms of Euler parameters.

Replacing Euler Angles with Euler Parameters

Consider a Keplerian motion of a satellite about a primary gravitational body with a center of mass at O. The orbit plane-fixed frame, $\mathcal{B}$, is a standard perifocal, Cartesian, dextral coordinate system centered at O. The inertial frame of reference, $\mathcal{I}$, is a standard Earth-centered Cartesian, dextral frame. The inertial Keplerian equations of the satellite's motion are given by the Newtonian relationship

$$\ddot{\boldsymbol{r}} = -(\mu/r^3)\boldsymbol{r} \tag{1}$$

where $\boldsymbol{r} \in \mathbb{R}^3\backslash\{\boldsymbol{0}\}$ is the inertial position vector and μ is the gravitational constant. The magnitude of r is given by the polar conic equation

$$r = \|\boldsymbol{r}\| = \frac{a(1-e^2)}{(1+e\cos f)} \tag{2}$$

where a, e, f are the semimajor axis, eccentricity, and true anomaly, respectively.

The nonlinear equations of motion, Eq. (1), can be straightforwardly solved for $\boldsymbol{r}$ and $\boldsymbol{v} = \dot{\boldsymbol{r}}$ using a transformation from $\mathcal{B}$ to $\mathcal{I}$ expressed by the directional cosines matrix (DCM)[10]:

$$T(\omega, i, \Omega) = \begin{bmatrix} c_\Omega c_\omega - s_\Omega s_\omega c_i & -c_\Omega s_\omega - s_\Omega c_\omega c_i & s_\Omega s_i \\ s_\Omega c_\omega + c_\Omega s_\omega c_i & -s_\Omega s_\omega + c_\Omega c_\omega c_i & -c_\Omega s_i \\ s_\omega s_i & c_\omega s_i & c_i \end{bmatrix} \tag{3}$$

where Ω is the right ascension of the ascending node, i is the inclination, ω is the argument of periapsis, and we have used the compact notation $s_x = \sin(x)$, $c_x = \cos(x)$.

The resulting inertial solutions are

$$\boldsymbol{r} = \frac{a(1-e^2)}{1+e\cos f}\begin{bmatrix} \cos(f+\omega)\cos\Omega - \cos i\sin(f+\omega)\sin\Omega \\ \cos i\cos\Omega\sin(f+\omega) + \cos(f+\omega)\sin\Omega \\ \sin i\sin(f+\omega) \end{bmatrix}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \sqrt{\frac{\mu}{a(1-e^2)}}\begin{bmatrix} -\cos\Omega\sin(f+\omega) - \sin\Omega\cos i\cos(f+\omega) - e(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i) \\ \cos\Omega\cos i\cos(f+\omega) - \sin\Omega\sin(f+\omega) - e(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i) \\ \sin i(\cos(f+\omega) + e\cos\omega) \end{bmatrix} \tag{4}$$

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*Senior Lecturer, Faculty of Aerospace Engineering. Senior Member AIAA.

Thus, the inertial position and velocity depend on time and the classical orbital elements, $[a, e, i, \Omega, \omega, M_0]^T$, where M_0 is the mean anomaly at epoch.

However, the preceding procedure introduces an inherent singularity attributable to the use of Euler angles to carry out the rotation from $\mathcal{B}$ to $\mathcal{I}$. This singularity manifests itself in the variational equations, comprising the rates of the Euler angles ω, i, Ω. A natural and physically meaningful methodology to avoid the singularities is to use Euler parameters instead of Euler angles to perform the rotation. To this end, let a unit quaternion be given by $\boldsymbol{q} = [\epsilon_4, \boldsymbol{\epsilon}^T]^T \in \mathbb{H}$. The elements of $\boldsymbol{\epsilon}$ are the Euler parameters, $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, \epsilon_3]^T \in \mathbb{S}^3$, satisfying the constraint

$$\epsilon_4^2 + \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = 1 \tag{5}$$

Given a single rotation of magnitude ϕ about the Euler vector, the relationship between the Euler parameters and the rotation angle ϕ can be written as[9]

$$\epsilon_1 = \sin(\phi/2)\cos\phi_1 \tag{6}$$

$$\epsilon_2 = \sin(\phi/2)\cos\phi_2 \tag{7}$$

$$\epsilon_3 = \sin(\phi/2)\cos\phi_3 \tag{8}$$

where ϕ_i is the angle between $\boldsymbol{\phi}$ and the inertial unit vectors, and

$$\epsilon_4 = \cos(\phi/2) \tag{9}$$

The DCM rotating from $\mathcal{B}$ to $\mathcal{I}$ can now be rewritten in terms of Euler parameters instead of Euler angles[9]:

$$T(\epsilon_1, \epsilon_2, \epsilon_3) = \begin{bmatrix} \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2 + \epsilon_4^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & \epsilon_2^2 + \epsilon_4^2 - \epsilon_1^2 - \epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & \epsilon_3^2 + \epsilon_4^2 - \epsilon_1^2 - \epsilon_2^2 \end{bmatrix} \tag{10}$$

Using the new DCM (10) to transform from $\mathcal{B}$ to $\mathcal{I}$, the expressions for the inertial position and velocity become

$$\boldsymbol{r} = T(\epsilon_1, \epsilon_2, \epsilon_3)\boldsymbol{r}_{\mathcal{B}}(a, e, M_0, t) = \boldsymbol{r}(a, e, \epsilon_1, \epsilon_2, \epsilon_3, M_0, t)$$

$$= \frac{a(1-e^2)}{1+e\cos f} \times \begin{bmatrix} \left(\epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2 + \epsilon_4^2\right)\cos f + 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4)\sin f \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4)\cos f + \left(\epsilon_2^2 + \epsilon_4^2 - \epsilon_1^2 - \epsilon_3^2\right)\sin f \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4)\cos f + 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4)\sin f \end{bmatrix} \tag{11}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = T(\omega, i, \Omega)\boldsymbol{v}_{\mathcal{B}}(a, e, M_0, t) = \boldsymbol{v}(a, e, i, \Omega, \omega, M_0, t)$$

$$= \sqrt{\frac{\mu}{a(1-e^2)}} \times \begin{bmatrix} 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4)\cos f + \left(2\epsilon_2^2 + 2\epsilon_3^2 - 1\right)\sin f + 2e(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) \\ \left(1 - 2\epsilon_1^2 - 2\epsilon_3^2\right)\cos f - 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4)\sin f - e\left(2\epsilon_1^2 + 2\epsilon_3^2 - 1\right) \\ 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4)\cos f + 2(\epsilon_2\epsilon_4 - \epsilon_1\epsilon_3)\sin f + 2e(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) \end{bmatrix} \tag{12}$$

Thus, we have obtained a parameterization of the motion using the new set of orbital elements

$$\boldsymbol{\alpha} = [a, e, \epsilon_1, \epsilon_2, \epsilon_3, M_0]^T \tag{13}$$

We can now obtain the nonsingular variational equations using these new constants of the motion.

Nonsingular Variational Equations

Lagrange's planetary equations (LPEs) are variational equations describing the time change of the orbital elements due to a potential perturbation. Written using the classical orbital elements, these equations become singular for equatorial ($i = 0$) and/or circular ($e = 0$) orbits. The most common method to deal with these singularities is to redefine the set of orbital elements. Thus far, the most common approach has relied on equinoctial elements.[2,5] We shall perform an alternative derivation using Euler parameters.

The LPEs, written in terms of osculating orbital elements, assume the compact form

$$\dot{\boldsymbol{\alpha}} = L^{-1}\left[\frac{\partial R}{\partial \boldsymbol{\alpha}}\right]^T \tag{14}$$

where $R = R(\boldsymbol{\alpha})$ is the perturbation potential and L is the skew-symmetric Lagrange matrix, found from

$$L = \left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\alpha}}\right]^T\left[\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\alpha}}\right] - \left[\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\alpha}}\right]^T\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\alpha}}\right] \tag{15}$$

The entries of the Lagrange matrix are the Lagrangian brackets,

$$[\alpha_i, \alpha_j] = \left[\frac{\partial \boldsymbol{r}}{\partial \alpha_i}\right]^T\left[\frac{\partial \boldsymbol{v}}{\partial \alpha_j}\right] - \left[\frac{\partial \boldsymbol{r}}{\partial \alpha_j}\right]^T\left[\frac{\partial \boldsymbol{v}}{\partial \alpha_i}\right] \tag{16}$$

satisfying[10]

$$[\alpha_i, \alpha_i] = 0, \qquad [\alpha_i, \alpha_j] = -[\alpha_j, \alpha_i], \qquad \frac{\partial[\alpha_i, \alpha_j]}{\partial t} = 0 \tag{17}$$

To calculate the Lagrangian brackets, it is convenient to rewrite the perifocal position $\boldsymbol{r}_{\mathcal{B}}$ and the perifocal velocity $\boldsymbol{r}_{\mathcal{B}}$ using the eccentric anomaly E as the independent variable[10]:

$$\boldsymbol{r}_{\mathcal{B}} = \begin{bmatrix} a(\cos E - e) \\ b\sin E \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_{\mathcal{B}} = \begin{bmatrix} -\dfrac{an\sin E}{1 - e\cos E} \\ \dfrac{bn\cos E}{1 - e\cos E} \\ 0 \end{bmatrix} \tag{18}$$

where $b = a\sqrt{(1-e^2)}$ is the semiminor axis and $n = \sqrt{(\mu/a^3)}$ is the mean motion. The partial derivatives of the position and velocity with respect to the Euler parameters are

$$\frac{\partial \boldsymbol{r}}{\partial \epsilon_i} = \frac{\partial T(\epsilon_1, \epsilon_2, \epsilon_3)}{\partial \epsilon_i}\boldsymbol{r}_{\mathcal{B}}, \qquad \frac{\partial \boldsymbol{v}}{\partial \epsilon_i} = \frac{\partial T(\epsilon_1, \epsilon_2, \epsilon_3)}{\partial \epsilon_i}\boldsymbol{v}_{\mathcal{B}}, \qquad i = 1, 2, 3 \tag{19}$$

By using the expression for the DCM given in Eq. (10), we find that

$$\frac{dT}{d\epsilon_1} = \begin{bmatrix} 0 & 2\epsilon_2 + 2\epsilon_3\epsilon_1/\epsilon_4 & 2\epsilon_3 - 2\epsilon_2\epsilon_1/\epsilon_4 \\ 2\epsilon_2 - 2\epsilon_3\epsilon_1/\epsilon_4 & -4\epsilon_1 & -2\epsilon_4 + 2\epsilon_1^2/\epsilon_4 \\ 2\epsilon_3 + 2\epsilon_2\epsilon_1/\epsilon_4 & 2\epsilon_4 - 2\epsilon_1^2/\epsilon_4 & -4\epsilon_1 \end{bmatrix} \tag{20}$$

$$\frac{dT}{d\epsilon_2} = \begin{bmatrix} -4\epsilon_2 & 2\epsilon_1 + 2\epsilon_2\epsilon_3/\epsilon_4 & 2\epsilon_4 - 2\epsilon_2^2/\epsilon_4 \\ 2\epsilon_1 - 2\epsilon_2\epsilon_3/\epsilon_4 & 0 & 2\epsilon_3 + 2\epsilon_1\epsilon_2/\epsilon_4 \\ -2\epsilon_4 + 2\epsilon_2^2/\epsilon_4 & 2\epsilon_3 - 2\epsilon_2\epsilon_1/\epsilon_4 & -4\epsilon_2 \end{bmatrix} \tag{21}$$

$$\frac{dT}{d\epsilon_3} = \begin{bmatrix} -4\epsilon_3 & -2\epsilon_4 + 2\epsilon_3^2/\epsilon_4 & 2\epsilon_1 - 2\epsilon_2\epsilon_3/\epsilon_4 \\ 2\epsilon_4 - 2\epsilon_3^2/\epsilon_4 & -4\epsilon_3 & 2\epsilon_2 + 2\epsilon_1\epsilon_3/\epsilon_4 \\ 2\epsilon_1 + 2\epsilon_3\epsilon_2/\epsilon_4 & 2\epsilon_2 - 2\epsilon_1\epsilon_3/\epsilon_4 & 0 \end{bmatrix} \tag{22}$$

The calculations may be considerably simplified if we utilize the time-invariance property of the Lagrangian brackets [cf. Eqs. (17)], implying that the brackets can be evaluated anywhere along the orbit. A natural choice is the periapsis, where $E = 0$, so that

$$\boldsymbol{r}_{\mathcal{B}} = \begin{bmatrix} a(1-e) \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_{\mathcal{B}} = \begin{bmatrix} 0 \\ \dfrac{bn}{1-e} \\ 0 \end{bmatrix} \tag{23}$$

Substituting Eqs. (20–22) and (23) into Eqs. (27) yields

$$\frac{\partial \boldsymbol{r}}{\partial \epsilon_1} = a(1-e)\begin{bmatrix} 0 \\ 2(\epsilon_2 - \epsilon_3\epsilon_1/\epsilon_4) \\ 2(\epsilon_3 + \epsilon_2\epsilon_1/\epsilon_4) \end{bmatrix}$$

$$\frac{\partial \boldsymbol{v}}{\partial \epsilon_1} = \frac{bn}{1-e}\begin{bmatrix} 2(\epsilon_2 + \epsilon_1\epsilon_3/\epsilon_4) \\ -4\epsilon_1 \\ 2\left(\epsilon_4 - \epsilon_1^2/\epsilon_4\right) \end{bmatrix} \tag{24}$$

$$\frac{\partial \boldsymbol{r}}{\partial \epsilon_2} = a(1-e)\begin{bmatrix} -4\epsilon_2 \\ (2\epsilon_1 - \epsilon_2\epsilon_3/\epsilon_4) \\ 2\left(-\epsilon_4 + 2\epsilon_2^2/\epsilon_4\right) \end{bmatrix}$$

$$\frac{\partial \boldsymbol{v}}{\partial \epsilon_2} = \frac{bn}{1-e}\begin{bmatrix} 2(\epsilon_1 + \epsilon_2\epsilon_3/\epsilon_4) \\ 0 \\ 2(\epsilon_3 - \epsilon_1\epsilon_2/\epsilon_4) \end{bmatrix} \tag{25}$$

$$\frac{\partial \boldsymbol{r}}{\partial \epsilon_3} = a(1-e)\begin{bmatrix} -4\epsilon_3 \\ 2\left(\epsilon_4 - \epsilon_3^2/\epsilon_4\right) \\ 2(\epsilon_1 + \epsilon_2\epsilon_3/\epsilon_4) \end{bmatrix}$$

$$\frac{\partial \boldsymbol{v}}{\partial \epsilon_3} = \frac{bn}{1-e}\begin{bmatrix} 2\left(-\epsilon_4 + \epsilon_3^2/\epsilon_4\right) \\ -4\epsilon_3 \\ 2(\epsilon_2 - \epsilon_1\epsilon_3/\epsilon_4) \end{bmatrix} \tag{26}$$

To calculate the partial derivatives of the inertial position and velocity with respect to the remaining orbital elements, we again use the separation of in-plane motion and the orbital plane attitude. Let us denote $\beta_i = \{a, e, M_0\}$ for conciseness, so that at periapsis

$$\frac{\partial \boldsymbol{r}}{\partial \beta_i} = T(\epsilon_1, \epsilon_2, \epsilon_3)\left[\frac{\partial \boldsymbol{r}_{\mathcal{B}}}{\partial \beta_i} + \frac{\partial \boldsymbol{r}_{\mathcal{B}}}{\partial E}\frac{\partial E}{\partial \beta_i}\right]_{E=0}$$

$$\frac{\partial \boldsymbol{v}}{\partial \beta_i} = T(\epsilon_1, \epsilon_2, \epsilon_3)\left[\frac{\partial \boldsymbol{v}_{\mathcal{B}}}{\partial \beta_i} + \frac{\partial \boldsymbol{v}_{\mathcal{B}}}{\partial E}\frac{\partial E}{\partial \beta_i}\right]_{E=0} \tag{27}$$

It is straightforward to show that the following relationships hold at periapsis[10]:

$$\frac{\partial E}{\partial a} = -\frac{3n\tau}{2r}, \qquad \frac{\partial E}{\partial e} = 0, \qquad \frac{\partial E}{\partial M_0} = \frac{1}{1-e} \tag{28}$$

where τ is the epoch. Substitution of Eqs. (28) into Eqs. (27) by using DCM (10) yields

$$\frac{\partial \boldsymbol{r}}{\partial a} = \frac{1}{a^2(1-e)} \times \begin{Bmatrix} -\left[3bn\tau a(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) + a^2(1-e)^2\left(2\epsilon_2^2 + 2\epsilon_3^2 - 1\right)\right] \\ \left[3bn\tau a\left(2\epsilon_1^2 + 2\epsilon_3^2 - 1\right) + a^2(1-e)^2(4\epsilon_1\epsilon_2 + 4\epsilon_3\epsilon_4 - 1)\right]/2 \\ -\left[3bn\tau a(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) + a^2(1-e)^2(-2\epsilon_1\epsilon_3 + 2\epsilon_2\epsilon_4)\right] \end{Bmatrix} \tag{29}$$

$$\frac{\partial \boldsymbol{v}}{\partial a} = \frac{n}{a^2(1-e)^2} \times \begin{Bmatrix} -\left[2ba(1-e)(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) + 3a^2n\tau\left(2\epsilon_2^2 + 2\epsilon_3^2 - 1\right)\right]/2 \\ \left[ba(1-e)\left(2\epsilon_1^2 + 2\epsilon_3^2 - 1\right) + 6a^2n\tau\left(\epsilon_1\epsilon_2^2 + \epsilon_3\epsilon_4\right)\right]/2 \\ ba(1-e)(-2\epsilon_2\epsilon_3 - 2\epsilon_1\epsilon_4) + 3a^2n\tau(\epsilon_1\epsilon_3 - 3\epsilon_2\epsilon_4) \end{Bmatrix} \tag{30}$$

$$\frac{\partial \boldsymbol{r}}{\partial e} = a\begin{bmatrix} 2\epsilon_2^2 + 2\epsilon_3^2 - 1 \\ -2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) \\ -2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) \end{bmatrix}$$

$$\frac{\partial \boldsymbol{v}}{\partial e} = \frac{na^2}{b(1-e)}\begin{bmatrix} 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) \\ -\left(2\epsilon_1^2 + 2\epsilon_2^2 - 1\right) \\ 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) \end{bmatrix} \tag{31}$$

$$\frac{\partial \boldsymbol{r}}{\partial M_0} = \frac{b}{1-e}\begin{bmatrix} 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) \\ 1 - 2\epsilon_1^2 - 2\epsilon_3^2 \\ 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) \end{bmatrix}$$

$$\frac{\partial \boldsymbol{v}}{\partial M_0} = \frac{na}{(1-e)^2}\begin{bmatrix} 2\epsilon_2^2 + 2\epsilon_3^2 - 1 \\ -2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) \\ -2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) \end{bmatrix} \tag{32}$$

Ordering Eqs. (24–26) and (29–32) and substituting into Eq. (16) yields

$$[a, e] = 0, \qquad [a, \epsilon_1] = -bn\frac{\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4}{\epsilon_4}$$

$$[a, \epsilon_2] = bn\frac{\epsilon_1\epsilon_4 - \epsilon_2\epsilon_3}{\epsilon_4}, \qquad [a, \epsilon_3] = bn\frac{\epsilon_1^2 + \epsilon_2^2 - 1}{\epsilon_4}$$

$$[a, M_0] = -\frac{an}{2} \tag{33}$$

$$[e, \epsilon_1] = \frac{2ea^3n(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4)}{\epsilon_4 b}, \qquad [e, \epsilon_2] = -\frac{2ea^3n(\epsilon_1\epsilon_4 - \epsilon_2\epsilon_3)}{\epsilon_4 b}$$

$$[e, \epsilon_3] = -\frac{2ea^3n\left(\epsilon_1^2 + \epsilon_2^2 - 1\right)}{\epsilon_4 b}, \qquad [e, M_0] = 0 \tag{34}$$

$$[\epsilon_1, \epsilon_2] = 4abn, \qquad [\epsilon_1, \epsilon_3] = \frac{4abn\epsilon_1}{\epsilon_4}, \qquad [\epsilon_1, M_0] = 0 \tag{35}$$

$$[\epsilon_2, \epsilon_3] = \frac{4abn\epsilon_2}{\epsilon_4}, \qquad [\epsilon_2, M_0] = 0 \tag{36}$$

$$[\epsilon_3, M_0] = 0 \tag{37}$$

These equations form the entries of the Lagrange matrix. Due to the anticommutativity of the Lagrangian brackets [cf. Eqs. (17)], only 15 distinct brackets are required.

The LPEs are now obtained by inverting the Lagrange matrix and substituting into Eq. (14). This procedure yields the following LPEs:

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial M_0} \tag{38}$$

$$\dot{e} = \frac{\sqrt{1-e^2}}{2a^2ne}\left(-\epsilon_2\frac{\partial R}{\partial \epsilon_1} + \epsilon_1\frac{\partial R}{\partial \epsilon_2} - \epsilon_4\frac{\partial R}{\partial \epsilon_3} + 2\sqrt{1-e^2}\frac{\partial R}{\partial M_0}\right) \tag{39}$$

$$\dot{\epsilon}_1 = \frac{1}{4a^2n}\left(\frac{2\sqrt{1-e^2}\epsilon_2}{e}\frac{\partial R}{\partial e} + \frac{\epsilon_1^2 + \epsilon_2^2 - 1}{\sqrt{1-e^2}}\frac{\partial R}{\partial \epsilon_2} + \frac{\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4}{\sqrt{1-e^2}}\frac{\partial R}{\partial \epsilon_3}\right) \tag{40}$$

$$\dot{\epsilon}_2 = \frac{1}{4a^2n}\left(-\frac{2\sqrt{1-e^2}\epsilon_1}{e}\frac{\partial R}{\partial e} - \frac{\epsilon_1^2+\epsilon_2^2-1}{\sqrt{1-e^2}}\frac{\partial R}{\partial \epsilon_1} - \frac{\epsilon_2\epsilon_4+\epsilon_1\epsilon_3}{\sqrt{1-e^2}}\frac{\partial R}{\partial \epsilon_3}\right) \tag{41}$$

$$\dot{\epsilon}_3 = \frac{1}{4a^2n}\left(\frac{2\sqrt{1-e^2}\epsilon_4}{e}\frac{\partial R}{\partial e} + \frac{\epsilon_1\epsilon_4-\epsilon_2\epsilon_3}{\sqrt{1-e^2}}\frac{\partial R}{\partial \epsilon_1} + \frac{\epsilon_2\epsilon_4+\epsilon_1\epsilon_3}{\sqrt{1-e^2}}\frac{\partial R}{\partial \epsilon_2}\right) \tag{42}$$

$$\dot{M}_0 = -\frac{2}{an}\frac{\partial R}{\partial a} - \frac{1-e^2}{a^2ne}\frac{\partial R}{\partial e} \tag{43}$$

Comparing to the classical LPEs, we see that, although the singularity at $i=0$ has been removed, as expected, due to the use of Euler parameters (it is well known that Euler angles rates are singular at the angle of the second rotation, i in our case), the singularity at $e=0$ remains. However, this singularity can also be removed by using the eccentricity measure

$$\eta = \sqrt{1-e^2} \tag{44}$$

as an orbital element instead of e. The transformations

$$\frac{\partial R}{\partial e} = \frac{\partial R}{\partial \eta}\frac{d\eta}{de} = -\frac{e}{\sqrt{1-e^2}}\frac{\partial R}{\partial \eta} \tag{45}$$

$$\dot{\eta} = \frac{d\eta}{de}\dot{e} = -\frac{e}{\sqrt{1-e^2}}\dot{e} \tag{46}$$

yield the LPEs

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial M_0} \tag{47}$$

$$\dot{\eta} = \frac{1}{2a^2n}\left(\epsilon_2\frac{\partial R}{\partial \epsilon_1} - \epsilon_1\frac{\partial R}{\partial \epsilon_2} + \epsilon_4\frac{\partial R}{\partial \epsilon_3} - 2\eta\frac{\partial R}{\partial M_0}\right) \tag{48}$$

$$\dot{\epsilon}_1 = \frac{1}{4a^2n}\left(-2\epsilon_2\frac{\partial R}{\partial \eta} + \frac{\epsilon_1^2+\epsilon_2^2-1}{\eta}\frac{\partial R}{\partial \epsilon_2} + \frac{\epsilon_2\epsilon_3-\epsilon_1\epsilon_4}{\eta}\frac{\partial R}{\partial \epsilon_3}\right) \tag{49}$$

$$\dot{\epsilon}_2 = \frac{1}{4a^2n}\left(2\epsilon_1\frac{\partial R}{\partial \eta} - \frac{\epsilon_1^2+\epsilon_2^2-1}{\eta}\frac{\partial R}{\partial \epsilon_1} - \frac{\epsilon_2\epsilon_4+\epsilon_1\epsilon_3}{\eta}\frac{\partial R}{\partial \epsilon_3}\right) \tag{50}$$

$$\dot{\epsilon}_3 = \frac{1}{4a^2n}\left(-2\epsilon_4\frac{\partial R}{\partial \eta} + \frac{\epsilon_1\epsilon_4-\epsilon_2\epsilon_3}{\eta}\frac{\partial R}{\partial \epsilon_1} + \frac{\epsilon_2\epsilon_4+\epsilon_1\epsilon_3}{\eta}\frac{\partial R}{\partial \epsilon_2}\right) \tag{51}$$

$$\dot{M}_0 = -\frac{2}{an}\frac{\partial R}{\partial a} + \frac{\eta}{a^2n}\frac{\partial R}{\partial \eta} \tag{52}$$

which are regular for noncollision orbits ($e \neq 1$). These LPEs constitute a simpler alternative to the equinoctial LPEs[10] and can be used to analyze the perturbed dynamics of circular equatorial orbits both analytically and numerically.

Example: Zonal Gravitational Harmonics

We shall illustrate the use of the newly derived variational equations by considering perturbations due to zonal gravitational harmonics. The gravitational potential including zonal harmonics only is given by[10]

$$R_z = -\frac{\mu}{r}\sum_{k=2}^{\infty} J_k\left(\frac{r_{\text{eq}}}{r}\right)^k P_k(\cos\phi) = -\sum_{k=2}^{\infty} R_{J_k} \tag{53}$$

where ϕ is the colatitude angle, satisfying

$$\cos\phi = \boldsymbol{r}\boldsymbol{s}_3/r \tag{54}$$

P_k is a Legendre polynomial of the first kind of order k, and r_{eq} is the primary's equatorial radius.

It is straightforward to express Eq. (53) in terms of Euler parameters, based on relationship (54). By recognizing that Eq. (54) is a direction cosine of the inertial position vector, we immediately find from Eq. (11) that

$$\cos\phi = 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4)\cos f + 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4)\sin f \tag{55}$$

For comparison, in terms of Euler angles, the colatitude angle can be calculated based on Eq. (4), yielding

$$\cos\phi = \sin i \sin(f+\omega) \tag{56}$$

The most dominant term in Eq. (53) is the J_2 term. The second-order Legendre polynomial satisfies $P_2(x) = 3x^2 - 1$. Therefore, we may write an expression for the J_2 perturbing potential, which reads

$$R_{J_2} = -\frac{\mu J_2 r_{\text{eq}}^2}{2p^3}(1+e\cos f)^3\left\{12[(\epsilon_1\epsilon_3-\epsilon_2\epsilon_4)\cos f + (\epsilon_2\epsilon_3+\epsilon_1\epsilon_4)\sin f]^2 - 1\right\} \tag{57}$$

where $p = a\eta^2$ is the semilatus rectum. This expression can be substituted into LPEs (47–52) to yield the effect of J_2.

To capture the secular effects only, R_{J_2} may be averaged as follows:

$$\bar{R}_{J_2} = \frac{n}{2\pi h}\int_0^{2\pi} R_{J_2} r^2 \,\mathrm{d}f \tag{58}$$

where h is the orbital angular momentum. In terms of the new orbital elements, the secular potential becomes (we shall omit the bar notation from the following elements)

$$\bar{R}_{J_2} = \frac{J_2 n^2 r_{\text{eq}}^2\left[1 - 6\epsilon_1^2 - 6\epsilon_2^2 + 6\left(\epsilon_1^2+\epsilon_2^2\right)^2\right]}{2\eta^3} \tag{59}$$

which is independent of ϵ_3.

Substituting into Eqs. (47–52) yields the first-order averaged Euler parameter-based planetary equations for an oblateness-perturbed satellite:

$$\dot{a} = 0 \tag{60}$$

$$\dot{\eta} = 0 \tag{61}$$

$$\dot{\epsilon}_1 = \tfrac{3}{4}J_2(r_{\text{eq}}/p)^2 n\epsilon_2\left[3 + 10\left(\epsilon_1^2+\epsilon_2^2\right)^2 - 12\left(\epsilon_1^2+\epsilon_2^2\right)\right] \tag{62}$$

$$\dot{\epsilon}_2 = -\tfrac{3}{4}J_2(r_{\text{eq}}/p)^2 n\epsilon_1\left[3 + 10\left(\epsilon_1^2+\epsilon_2^2\right)^2 - 12\left(\epsilon_1^2+\epsilon_2^2\right)\right] \tag{63}$$

$$\dot{\epsilon}_3 = \tfrac{3}{4}J_2(r_{\text{eq}}/p)^2 n\epsilon_4\left[1 + 10\left(\epsilon_1^2+\epsilon_2^2\right)^2 - 8\left(\epsilon_1^2+\epsilon_2^2\right)\right] \tag{64}$$

$$\dot{M}_0 = \tfrac{3}{2}J_2(r_{\text{eq}}/p)^2 n\eta\left[1 - 6\epsilon_1^2 - 6\epsilon_2^2 + 6\left(\epsilon_1^2+\epsilon_2^2\right)^2\right] \tag{65}$$

For example, if we consider an equatorial orbit, $i=0$, then $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_4 = \sqrt{(1-\epsilon_3^2)}$, and the averaged differential equation for the Euler parameter ϵ_3 becomes

$$\dot{\epsilon}_3 = \tfrac{3}{4}J_2(r_{\text{eq}}/p)^2 n\sqrt{1-\epsilon_3^2} \tag{66}$$

which is readily solvable for the initial condition $\epsilon_3(0)$:

$$\epsilon_3 = \epsilon_3(0)\cos(ct) + \sqrt{1-\epsilon_3(0)}\sin(ct) \tag{67}$$

where

$$c = \tfrac{3}{4}J_2(r_{\text{eq}}/p)^2 n \tag{68}$$

Examining relationships (6–8), we can interpret the result geometrically. Since $\epsilon_1 = \epsilon_2 = 0$, then $\phi_1 = \phi_2 = \pi/2$, implying that the Euler vector coincides with the inertial $\boldsymbol{s}_3$ axis, and the long-term orbital spin motion is about the north pole, as expected, so that $\phi_3 = 0$, and

$$\sin(\phi/2) = \epsilon_3(0)\cos(ct) + \sqrt{1-\epsilon_3(0)}\,\sin(ct)$$
$$= \sin\left[ct + \sin^{-1}\epsilon_3(0)\right] \tag{69}$$

Solving for ϕ gives

$$\phi = 2ct + 2\sin^{-1}\epsilon_3(0) \tag{70}$$

Thus, we recover the known fact that J_2-perturbed equatorial orbits spin about the inertial $\boldsymbol{s}_3$ axis at rate $2c$. This observation agrees with the classical elements, which give, for $i = 0$,

$$\dot{\Omega} + \dot{\omega} = 2c = \dot{\phi} \tag{71}$$

Conclusions

This Note showed that Euler parameters may be used as a benign set of orbital elements replacing the classical elements. Compared to other nonsingular elements such as the equinoctial elements, the Euler parameters have a clear physical and geometrical interpretation, yielding variational equations which are less complex than the equinoctial equations.

We also showed that the representation of a zonal gravitational potential using the new elements is straightforward, and we derived an averaged set of J_2-perturbed equations using the new orbital elements.

We conclude that Euler parameters show much promise for astrodynamic and astronomical applications, constituting a worthy alternative to the classical elements-based nonsingular elements.

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